Modes of reasoning in the mathematics classroom: a comparative investigation

Eva Jablonka
King's College London

This paper attempts to map a range of modes of mathematical reasoning employed in classrooms from Germany, Hong Kong and the United States taught by experienced teachers locally judged to be competent. Reasoning here is used as an umbrella term for modes of justification within a range of strategies that aim at making discursively available some elements of mathematical practice. The significance of this analysis consists in the attempt of describing modes of reasoning in a way that accommodates the diversity of mathematical topics, achievement levels, curriculum traditions and culturally sanctioned modes of interaction, rather than in the outcome of the comparison itself.

Keywords: Reasoning, justification, comparison, discursive analysis

Introduction

In mathematics education, as well as in philosophy and history of mathematics, a range of notions has been employed to capture different modes of mathematical reasoning. Mathematical reasoning (judged by the standards of Greek mathematics) has been characterised as being based on clear and concise language and appreciation of logical inference for deriving conclusions, which is directed towards extinguishing subjective elements from judgments. Often a distinction is being made between the context of discovery and the context of justification in mathematical activity, and the relevance of non-deductive modes for the context of discovery, such as inductive and abductive reasoning, computational and visual evidence, evidence from measurements, and other heuristics, has been pointed out. Further, explanations have been distinguished from proofs. While some proofs at the same time function as explanations, the latter are often associated with the context of discovery, where they might motivate new definitions. Hence, motivating definitions (or axioms) is described as an activity of justifying that differs from providing a mathematical proof. Further, mostly within mathematics education, a distinction has been made between argumentation and proof, while the latter is seen as a special version of the first with restrictions on legitimate warrants, expression and type of inferences (Boero, Douek and Ferrari, 2008; Pedemonte, 2008).

Curriculum frameworks usually include statements about some forms of mathematical reasoning, which students are expected to acquire. Related learning expectations pertain to a range of activities, such as scrutinising and justifying results of operations, explaining and motivating an approach to a problem, vindicating conjectures, verifying hypotheses, or justifying the validity of statements by local deductions and proofs. These activities are associated with developing students' understanding, as all are meant to include acts of interpretation and elaboration of meanings, which by most educators are ranked high and often contrasted with operationalised mathematical activities.

The task of the teacher in a particular classroom is to initiate students into what counts as mathematical practice, including its discursive and non-discursive elements. Providing reasons in the course of expositions ('explaining') is used by teachers as a pedagogic strategy for making discursively available to students some features of the entities and operations they are supposed to learn. As the principles for constructing mathematical arguments, or what is accepted as such in a particular classroom, cannot be made fully discursively explicit, students are expected to learn how to reason mathematically through participation and engagement. Hence, giving reasons as a pedagogic strategy aims at achieving both, modelling some forms of mathematical argument as well as rendering discursively available some principles of mathematical practice.

When and how students are expected to engage in reasoning is subject to the (emerging) norms for mathematical activities, which in the classroom intermingle with other social norms (cf. Bauersfeld, 1980; Yackel and Cobb, 1996). The exploration of reasoning activities in a range of classrooms from different countries promises to reveal how these might be shaped by culturally sanctioned forms of interaction, role-related asymmetries and pedagogical principles. The data used in this paper are from the Learner's Perspective Study and were recorded some years ago. The significance of this re-analysis consists in the methodological challenge of describing modes of reasoning in a way that accommodates the diversity of mathematical topics, achievement levels, curriculum traditions, rather than in the outcome of the comparison itself.

In this paper the focus is on students' contributions rather than on the teachers' explanations. The classrooms studied differed in average students' achievement, and it could be expected that achievement differences would be associated with some differences in students' ways of reasoning.

Methodology

In order to construct an empirically based language for describing the modes of reasoning, 'reasoning episodes' were identified in altogether 60 lessons (ten lessons from two classrooms in each country/region). An episode qualified as involving reasoning, if a 'reasoning exchange' was a part of the conversation. The person who provides a reason might interpret something as being not evident, doubtful or disputable (*prophylactic reasoning*), or is requested by another person (who interprets something as not evident, doubtful or disputable) to give a reason (*reasoning on request*). The episodes include the moves that prompted the reasoning, if any, and the 'closure' (e.g. signs of agreement or acceptance) and represent units of conversation with thematic coherence. The attempt to increase evidence or acceptance has to be visible for the other participants in the conversation, which can be seen by their reaction. These 'reasoning episodes' were identified in the transcripts and if needed the videos were consulted.

Student reasoning and prompts for uttering reasons

In the course of activities aimed at making discursively available things that are otherwise tacitly assumed or done (e.g. calculating, solving problems, producing graphs, drawing geometrical shapes) occasionally interpretations and elaborations of mathematical meanings, that is, *reasons*, were provided. As has been pointed out above, on the side of the teacher this is a pedagogic strategy. As far as the students are concerned, they did not frequently utter reasons for what they were saying or doing.
The table below shows how often students in each classroom (in 10 consecutive lessons) provided some mathematics-related reasons, either on request by their

teacher or on their own initiative. As not all conversations between students were captured in the video, especially during seatwork when several groups of students talked to each other, the count only includes reasoning episodes during whole class interaction. In all classrooms except the one from German School 1 (G1), students' explanations were more frequently produced on teachers' requests than prophylactically. As *argumentation* entails justification of an issue at stake by all involved, students cannot be expected to engage in argumentation with their teachers. However, there was a long episode where this occurred in G1, which accounts for the relatively high number of student-initiated reasoning episodes in this classroom. The students had been asked to 'prove' simple binomial expansions by means of a geometrical interpretation and present the outcome to their peers. The teacher happened to be unprepared for one group's line of argument and hence an 'a-didactical situation' (Brousseau, 1986) emerged.

Classroom, no. of students, achievement level	On teacher's request	On own initiative*
German School 1 (G1), 27, average-high	23	24
German School 3 (G3), 12, low-average	26	16
Hong Kong School 1 (HK1), 35, high- average	9	4
Hong Kong School 3 (HK3), 39, high	3	5
United States School 1 (US1), 29, low	82	10
United States School 2 (US2), 33, high	13	12

Table 1: Number of student reasoning episodes in 10 consecutive lessons from each classroom.

The high number of teacher-initiated student reasoning in US1 appeared to be due to an established norm that students 'explain their thinking'. In the other classrooms, teachers' requests for reasons, such as through asking, "Why is it …?" or, "How do you know …?" in most cases were to be taken as an indication that a student had produced something wrong. For example, many of the teacher's requests to students for backing up their solution procedures in G3 occurred in situations when the students' lack of familiarity with some basic arithmetic operations became obvious and the teacher wanted to elicit students' erroneous strategies. In contrast, the teachers in the Hong Kong classrooms did not have to struggle with a lack of students' fluency in arithmetic or did not find it worthwhile to spend much time with eliciting students' strategies when computational errors were made. The relatively low number of student reasoning episodes in the Hong Kong classrooms also reflects the norms for interaction, especially the more controlled turn taking mechanism. Student self-initiated reasoning often included the backup of a claim that another student's (and in some cases the teacher's) solution procedure was faulty, or cases when students thought a task posed by the teacher cannot be solved. Due to differences in how the hierarchy between teacher and students was constituted, such claims were more often made in the German classrooms, most commonly in G3 that also had the weakest regulated turn taking mechanism, which could be adhered because of the low number of students.

Identifying modes of reasoning across topics

In the six classrooms, much of the time the participants engaged in carrying out mathematical operations. In these activities, if the operations were of a local nature (such as solving a particular task), teachers (or occasionally students) provided an account of how the solution was achieved, sometimes while writing on the board, that is, they were *documenting* what they are doing. Reasons in these *documenting* activities might be provided by means of reference to the generalised procedures that are applied in the particular case.

When working with a new form of presentation (e.g. graph of a function, or an algebraic expression), the activities often consisted in *naming* and *defining*. Defining also included attempts of generalising graphical and symbolic expressions from particular cases. Providing reasons in a defining activity could entail showing the fruitfulness of definitions through making methodological judgments about alternatives (Tappenden, 2005). In US1 there were indeed instances when the teacher (more or less successfully) attempted to achieve this.

Making discursively explicit a generalised operation (e.g. for solving simultaneous linear equations) entails *explicating* a set of steps to be followed in a range of similar cases. Providing grounds in explicating activities included stating the condition under which the procedure is valid or attempting to motivate single steps by means of a more general heuristic.

When dealing with generalised expressions (‘formulae’) or generalised diagrams (e.g. a geometric shape without any specific properties), the activity of making discursively available the mathematical meanings and interpreting their relations (after setting up interpretations through defining) was used as a form of *proofing*.

The examples below provide instances of these activities. They are only from episodes, in which students were involved in some form of reasoning, and not when the teacher gave explanations. The headings of the short extracts indicate the topics of the episodes.

Defining

US1 - Exponents

T Now does that quite explain, though, how three to the zero is one?

S Yes

T How so [Name of student]?

S Cos it's three to the zero - it's the same as three to the - to the negative zero

T Say that one more time…please.

S Okay… um

T Little louder

S Three to the zero. It's almost the same as um ... three to the negative zero.

Before this episode, they had gone over 3^2 and 3^{-2}, 3^3 and 3^{-3}, and 3^4 and 3^{-4} to demonstrate a reciprocal relationship, and the student obviously built on this in his argument. The teacher in this classroom frequently asked the students to elaborate on the meanings of symbolic expressions and to give reasons (e.g. of the definition of ‘fraction’ or of the meaning of ‘A divided by 0’). As the students had no access to the mathematical principles, these attempts were in many cases not very successful in terms of reaching agreement on a motivation for a definition. (Had they been derived, the ‘definition’ would constitute a local theorem.). In US2, discussing definitions was also a common activity, however with less student involvement in terms of eliciting reasons from them. The outcomes of these ‘definitions’ often were used as a starting point for the sorts of activities that were practiced later in the lesson. The teachers from the other classrooms either did not motivate definitions or they provided some motivations themselves.

Documenting

This activity was the most common in all classrooms during whole class interaction with student involvement. While practicing operations through solving sets of similar tasks, students occasionally stated what they were doing ('explaining the steps'), in particular when presenting a solution to the whole class. As outlined in the previous section, in cases of disfluencies in the smooth flow of proceeding, when some doubts were uttered, or when errors became visible, the teachers requested a justification. Further, teachers asked for reasons, when they wanted to alert students to a common source of errors, as in the following example:

<u>G1 – Expanding algebraic expressions</u>

T Right… in the first problem there was something to pay attention to… why?

S Er… that there's a minus sign in front of the first pair of brackets so that it is turned round then

The teacher's request for a reason here is used as a strategy for pointing out particular pitfalls when executing the operation of expanding expressions. On request, the student is stating the condition for applying the operation as well attempting to state the procedure.

<u>G3 - Finding parameters of linear equations from graphs</u>

T Five is the point of intersection that MX in front doesn't exist you see. Why is there no MX in our equation now in that last example?

S Because it's zero

T Because M equals zero but our point of intersection does exist

Here the reason includes an explication of an element of a general procedure for reading off parameters from graphs. This was intended to provide an introduction to developing such a general procedure, rather than practising it.

<u>G3 – Calculating surface area and volume of various solid figures</u>

T So what kind of unit of measurement comes after this?

S Decimeters

S Decimeters second

T Right…square centimeters…why just square centimeters?

S Because we're dealing with a surface

This instance is from a lesson, in which the students struggled with choosing the appropriate units for volume and surface area, and the teacher frequently asked for warranting their choices. The student might refer to the general 'rule', 'Area is measured in square units, volume is measured in cubic units'.

<u>US1 - Powers and the order of operations</u>

T Is the final solution to this going to be positive or negative? Raise your hand. Is it going to be positive or negative? Final solution if we calculated an answer ... is it going to be positive or negative?

S It's gonna be a negative.

T It's gonna be a negative? Why - why do you say that?

S Because it's four times four is sixteen and then it's a positive sixteen and then a positive times a negative equals a negative.

T A positive times a negative, okay. Did everybody hear what he was saying?

Before this episode, they had introduced the notions $12 = 12^1$, (-4) (-4) (-4) as $(–4)^3$ and 3^4 = (3)(3)(3)(3); the teacher preferred the use of '( )' for denoting multiplication.

Explicating

There were not many instances of students' involvement in discussing generalised operations. Only in the Hong Kong classrooms, general methods for solving systems of linear equations were introduced during the lessons that were recorded.

<u>HK3 - Solving linear equations in two unknowns using elimination</u>

S1 [to S2] It's the same no matter addition or subtraction… aren't they all addition? I used subtraction…this one is subtraction…this one C is subtraction

S2 (?) negative sign…then followed by a negative two and it'll become positive two...

S1 Of course not…we do not need to follow…

S2 If there is a negative sign when I multiply... do I need to multiply negative two?

S1 No…if you subtract a smaller value from a larger one you can use either addition or subtraction…you can choose among them…

This conversation can be seen as being about a general rather than a local procedure, as S1 asked about what 'all' of the examples have in common.

<u>HK1 - Solving linear equations in two unknowns using substitution</u>

T Why? Try to explain that…why do we choose the first one instead of the second one…why? [T points to a student to answer the question][S stands up]

T Why do we choose the first equation instead of the second one?

S It is easier [S asks his neighbor before answering the question]

This example can be read as pointing to a heuristic of choosing the variable for substitution. Heuristics are generalised procedures, even if they are not fully realised in discourse.

Proofing

<u>G1 - Evaluating binomial products by geometrical representations</u>

S1 Right…so there's the problem, so here this big this white square this is erm here A squared… because because quantity A so A to the power of two and then we've got this small square here so

S2 this is B times B so B to the power of two

S1 Exactly and now we want to erm take away here… now we take away from A squared…

S2 You take away the small area from the large area

S1 Exactly… and then this here should be the result and so we've already solved it (?)

S3 Push the board a little bit

S1 (?) and here the result is a little complicated now

In this lesson, several students discussed over a long period of time their proofs for formulae for three binomial products (for $(a + b)^2$, $(a - b)^2$ and $(a + b)(a - b)$, respectively), which they presented at the board without much teacher intervention. The short episode shows how they backed up their claims by reference to the diagram, which they had set up by naming and defining lengths and areas.

<u>US2 – Linear and direct linear relation</u>

T Direct variation…okay…lets put that down…direct variation…um… is this direct variation folks? [Points to another graph on white board].

S No.

T No. Why not?

S Because it doesn't pass through the origin-

T It does not pass through the origin. Okay? //Everybody got it?

In US2 the activities entailed more often some form of proofing than in other classrooms, usually through referring to graphs or diagrams. However, as can be seen from the number of 'student reasoning episodes' in the ten lessons, the activity did not often involve students.

Further observations and conclusions

As to similarities and differences, the analysis showed that *documenting* and *explicating* constitute common modes of activities that can be described across topics, and that these were very common in all classrooms. This is not to suggest that the same amount of time was devoted to these activities, as the lesson structures significantly differed in the classrooms. The lessons in US2 appeared to be closer than those in US1 to the lesson pattern reported by Stigler and Hiebert (1999) as typical for US classrooms, but how the activities were enacted does not match that description. Neither did the lessons in the German classrooms show the pattern reported as typical for German classrooms (Clarke et al., 2007), but the lessons in HK3 resembled that 'German' pattern. As there were quite long periods of individual seatwork in HK3, the number of students' reasoning episodes in whole class interaction is necessarily low.

With respect to the type of mathematical tasks, the two Hong Kong classrooms were more similar to each other than any other pair of classrooms, which is due to the fact that the topics in both classrooms were largely the same. Further, due to the differences in turn taking mechanisms, the amount of 'filtering' of students' contributions in whole class discussions varied, with least 'filtering' in US1 and most in HK1 and in G1. The extent to which the students shaped or acquiesced to the teacher's pre-defined lesson structure varied according to the strength of the teacher's control of the forms of interaction that constituted the structure. In this respect, G3 and HK3 showed some similarities, as in both classrooms deviations occurred.

Based on these observations, the amount of student reasoning in whole class interaction seemed to be a function of the turn taking mechanisms in interactional routines, rather than of authority relationships in general. In all classrooms except US1, and to some extent in US2, where 'explaining one's thinking' was part of the classroom norm, the asymmetry between prophylactic reasoning and reasoning on request was reminiscent of the same asymmetry in warranting norms of conduct.

These norms usually are not warranted prophylactically, but are likely to be substantiated when being contravened (e.g. when excusing oneself).

The framework for describing different modes of mathematical activities and the types of reasons provided for their justification seems to be viable across mathematical topics. But it does not include some activities that do not belong to the esoteric domain of school mathematics, such as mathematical modelling or activities that include evidence from measurement. The activities of *documenting*, *explicating* and *proofing* can be related to elements of Dowling's (2013: 16) three-dimensional scheme of the "modality of general and local esoteric domain apparatus", in which these activities would entail moves from non-discursive to discursive semiotic modes. However, this is not the case for what above has been described as *defining* by means of generalising particular instances of expressions. In addition, there are other pathways for moving from local examples to general cases, as for example by means of establishing general procedures from local ones, which would entail formalising. Hence, the framework needs to be further developed.

As only episodes with student involvement in whole class interaction were included, the modes of activities and reasoning observed cannot be taken as a signature of any of the classrooms. For this purpose, teacher talk would need to be analysed. A question to be asked is whether different modes operate in low- and high-achieving classrooms, or in teacher interaction with low- and high-achieving students in the same classroom.

References

Bauersfeld, H. (1980) Hidden dimensions in the so-called reality of a mathematics classroom. *Educational Studies in Mathematics,* 11(1), 23–41.

Boero, P., Douek, N. & Ferrari, P.L. (2008) Developing mastery of natural language. In English, L. (Ed.), *International Handbook of Research in Mathematics Education* (pp. 262-295). New York: Routledge.

Brousseau G. (1986) Fondements et methods de la didactique des mathématiques. *Recherches en didactique des mathématiques,* 7(2), 33-115.

Clarke, D., Mesiti, C., O'Keefe, C., Jablonka, E., Mok, I., Xu, L.H. & Shimizu, Y. (2007) Addressing the Challenge of Legitimate International Comparisons of Classroom Practice, *International Journal of Educational Research,* 46(5), 280-293.

Dowling, P. (2013) Social Activity Method: A fractal language for mathematics. *Mathematics Education Research Journal*, 25, 317-340.

Pedemonte, B. (2007) How can the relationship between argumentation and proof be analysed? *Educational Studies in Mathematics,* 66(1), 23-41.

Stigler, J.W. & Hiebert, J. (1999) *The teaching gap: best ideas from the world's teachers for improving education in the classroom*. New York, NY: Free Press.

Tappenden, J. (2005) Proof style and understanding in mathematics I: visualization, unification and axiom choice. In Mancosu, P., Jørgensen, K.F. & Pedersen, S.A. (Eds.), *Visualization, explanation and reasoning styles in mathematics* (pp. 147-214). Dordrecht, The Netherlands: Springer.

Yackel, E., & Cobb, P. (1996) Sociomathematical norms, argumentation and autonomy in mathematics. *Journal for Research in Mathematics Education,* 27(4), 458–77.