An exploration of primary student teachers' understanding of fractions

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The purpose of this study was to discover the individual and distinct ways in which each student teacher understands fractions and their strategies for working with them. A phenomenographical approach was adopted in order to provide insight into each student teacher's subject knowledge of fractions. This study involved detailed scrutiny of six self-selected small groups, which enabled a range of rich and honestly reflective data to be collected. Groups undertook two collaborative tasks involving the sequencing of fractions by magnitude, followed by reflective interviews. Each group also undertook a diagnostic interview, considering a range of questions, which they had ordered in terms of their perceived difficulty. A constructivist perspective was adopted giving students the opportunity to reconstruct their own understanding of fractions through the explanation and discussion of their existing ideas. A range of successful strategies was demonstrated, especially the use of mathematical anchors and the use of residual or gap thinking as a means of comparison. Improper fractions and reunitising were the main difficulties cited by many in the group. A common assumption was that there was a particular 'correct' method to be adopted. The study helps to identify misconceptions that can be addressed within teacher training.

Key Words: fractions, primary mathematics, student teacher subject knowledge, phenomenography

Introduction

This study developed from a professional and personal interest in the learning and teaching of mathematics, in particular fractions, and a desire to make it more effective. It is acknowledged that there has already been considerable research focusing on the difficulties encountered by pupils in primary and secondary schools in many countries, especially the UK, USA and Australia. There have been fewer studies that focused specifically on student teachers' understanding of fractions (for example: Ball, 1990; Miller, 2004; Domoney, 2002; Toluk-Ucar, 2009). This study focuses on thirteen student teachers' understanding of fractions and the related areas of mathematics. It considers the aspects in which they feel confident as well as those that they perceive as problematic. The purpose of this study was to discover the 'qualitatively distinct ways' (Steffe, 1996: 321) in which students understood fractions.

Research design and methodology

One of the underlying assumptions of this study was that the mathematical understanding of the students and their ability to share that understanding with their pupils is integral to their perceptions of themselves as mathematicians and teachers.

Aubrey (1997: 3) claims that, "If teaching involves helping others to learn then understanding the subject content to be taught is a fundamental requirement of teaching".

The main methods employed were the use of observed shared tasks followed by reflective discussions and diagnostic group interviews. It was intended that the student teachers would explore, explain and possibly reconstruct their own understanding of fractions. These methods involved the explanation and discussion of their existing ideas and a consideration of any elements, which possibly caused confusion. Apart from exploring and explaining their present understanding it provided opportunities for them to develop a more effective, relational understanding (Skemp, 1989) of fractions and its related areas of mathematics.

Through the use of collaborative tasks and reflective discussion, it was intended to mirror the process described by Carpenter and Lehrer (1999: 20) through which mathematical understanding is promoted; "constructing relationships, extending and applying mathematical knowledge, reflecting on the experience, articulating what one knows and making mathematical knowledge one's own". In this way the study adopted a constructivist perspective. This is the belief that "knowledge is actively constructed by the cognising subject, not passively received from the environment" (Von Glasersfeld (1989: 162) in Ernest (1991). Ernest (1993: 63) described a social constructivist theory of learning mathematics which suggests that "both social processes and individual sense making have central and essential parts to play in the learning of mathematics".

The use of phenomenography, which is "the empirical study which seeks to understand how individuals experience, apprehend, perceive, conceptualise or understand the world", (Marton 1994: 4424) provides a valuable means for understanding learning from a student's point of view. Phenomenography has been used in a range of mathematical studies that considered the learning of children (Neuman, 1997) and of adults (Asghari and Tall, 2005). Phenomenography "takes human experience as its subject matter" (Marton and Neuman, 1996: 315). The study is based on the underlying premise that although participants are all undertaking the same task, there will be a number of qualitatively different ways of experiencing or understanding the question or problem which can be observed and identified. Each participant brings his or her prior experience and learning to the task and this affects the way in which it will be undertaken.

The intention of this study was to discover the nature of these differences. A convenience sample of thirteen student volunteers participated in the study. The group was predominantly female aged between 21 and 26. The majority were in their second year of a BA(Hons) degree in Primary Education, there were also two pairs from the Primary Postgraduate course. Two collaborative tasks were observed, where the student teachers worked in self- selected groups of two or three. The first involved the sequencing according to magnitude of a series of fractions, percentages and decimals and matching their equivalents, which was intended to be introductory and provide a supportive start to the student teachers engagement with the study. A second task followed the same style but focused purely on fractions with the option of some further fractions to be included for those who felt confident. These tasks focused on the part-whole and the measurement context of fractions (Kieren, 1976). After each observed tasks there were reflective discussions considering their strategies and any areas that were perceived to be particularly difficult.

A series of diagnostic interviews was conducted with individuals and pairs. These interviews were based on a range of questions that considered the findings of the observed tasks and the research literature. There were 20 questions, a sample of the type of questions can be seen below with the rationale and source for each one.

Ordering and Magnitude of Fractions	
<u>Question 6</u> What fractions come between $\frac{2}{5}$ and $\frac{3}{5}$?	This question developed the idea of continuity and fraction density. It was intended as an extension of sequencing activities. The open nature of the question was intended to promote discussion. It was based on a KS2 National Curriculum Test question for level 4.
Unitising and Reunitising	
Question 3 X X X X X X = $\frac{3}{2}$ of the unit. How many is there in a unit? Question 11 These circles represent $\frac{3}{7}$ of a unit. How many is the whole unit? ● ● ● ● ● ● ● ● ● Question 12 These circles represent $\frac{3}{4}$ of a unit. How many is $\frac{2}{3}$ of the unit? ● ● ● ● ● ● ● ● ●	Unitising refers to the process of constructing chunks in terms of which to think about a given commodity. It is a subjective process (Lamon 2005: 78). It is a natural process and plays an important role in several processes needed to understand fractions especially in partitioning and in equivalence. The questions are based on examples from Cramer and Lesh (1988). Rational Number Knowledge of Pre-service Elementary Education Teachers. The progression of questions building on the slightly simpler version (question 3). This progression of the questions was based on the views of students undertaking the pilot.
Approximation of magnitude of fractions and benchmarking task using near equivalence to 1.	
Question 5 Which is the best estimate for $\frac{12}{13} + \frac{7}{8}$? a) 1 b) 2 c) 19 d) 21 (multiple choice)	Question used with US grade 5 children, by Mitchell (2004) as it had also been used in National assessments and earlier research studies. The use of multiple choice was intended to provide a greater level of discussion.

Figure1 A Sample of Questions from the Diagnostic Interviews

At the start of each interview the participants were asked to indicate the questions about which they felt most and least confident, these perceptions were used to structure their interview. This was intended to follow the phenomenographical approach and ensured that each interview followed an individual path depending on the student's choices and enabled them to begin with the questions with which they felt more confident. A questionnaire was also used to provide further information about each participant's qualifications and feelings about their own learning in mathematics. The videoing of these activities focusing on the table and the writing produced a huge amount of data which was reviewed and coded and cross tabulated to identify the student teacher's perceived strengths and areas of difficulty.

Findings

Areas of Strength

The student teachers demonstrated confidence in a range of areas. Some of which were as expected, for example, the use of percentages, finding fractions of quantities and the use of equivalent fractions, usually by continuing an established pattern. There was also a range of strategies that were demonstrated across both observed activities and within the answering of questions within the diagnostic interviews. One such successful strategy that was adopted by individuals in several groups, was the initial placing of the more common fractions and their equivalents to provide a structure. This reflected the use of *mathematical anchors* (Spinillo, 2004). These were often used as a guide and referred to as 'boundaries' or 'markers' by the participants. For example, in activity one when placing $\frac{2}{5}$ on the number line;

> **Iris:** Two fifths feels bigger than a quarter and it must be less than a half, because that would be two and a half fifths. So it goes in the middle but nearer the half.

This comparison, to the more accessible numbers, for example, $\frac{1}{2}$ or 1 is also referred to as *Benchmarking* by Clarke at al. (2008). The use of mathematical anchors for comparison reflected the use of a half, as an anchor; made by eight and nine year old children when adding fractions (Spinillo, 2004) where it was considered to further facilitate their understanding. This strategy was also sometimes combined with the use of residual thinking (Clarke et al., 2008) where a learner refers to the amount needed to make a fraction up to a more accessible number, usually one or, in the following case, a half. In this example it enabled Anne to deduce that $\frac{45}{80}$ was larger than $\frac{55}{100}$.

> **Anne:** They are both five away from a half, but five hundredths is equal to a twentieth, and five eightieths is the same as a sixteenth. One twentieth is smaller so it will be nearer to the half.

When further explanation was needed by her student colleague, she elaborated with, "That (pointing to $\frac{45}{80}$) is…(greater) because it is a half and a sixteenth and that is bigger than a half and a twentieth." This was an interesting and natural use of a mathematical anchor, where her prior knowledge of a half and its equivalents was used effectively to enable her to make a comparison between apparently more complex fractions. This strategy was employed by six of the thirteen students, in a variety of ways, within the sequencing activities to establish the magnitude of a fraction by comparing it with one that they felt more certain about.

There were also some very personal strategies and approaches adopted, one example of this was demonstrated in response to the following question;

> At the ferry port, one quarter of the passengers are travelling to France, one third are going to Germany, what fraction are travelling to Holland?

Gill had drawn a circular diagram but was uncertain about how to establish what the remaining piece might be.

> **Ellen:** Well, a quarter looks like this…. Like a clock and if you drew a third, then this bit is Holland…
> **Gill:** So that bit is quarter past… and….
> **Ellen:** I couldn't work it out straight away like this… (new drawing)
> Can you see the 5 minutes… round the clock? $\frac{1}{12}$ and $\frac{2}{12}$ (pointing to each 5 minute section). Does that makes sense? $\frac{1}{4}$=$\frac{3}{12}$ and $\frac{1}{3} = \frac{4}{12}$ so that is $\frac{5}{12}$, it looks like 25 to…
>
> **Gill:** Hmmm, yes
> **Ellen**: I only know 'cos my dad used to teach me fractions on the clock.

Such moments within the diagnostic interviews proved enlightening as the interviewee took on the role of teacher and in sharing their thinking with their student colleague gave an insight into the strategies that they used to answer the question. A range of contrasting methods were employed, for example, Lynne favoured considering all fractions as decimals and had a wide range of what were described as 'known facts' such as $\frac{1}{9}$ is $0.\dot{1}$ and $\frac{1}{8}$ is 0.125 which enabled her to convert most fractions into 'something like a real number'. Fractions were considered too difficult to deal with so this was an approach developed at secondary school, which was then applied to most questions asked; it was mostly successful.

Areas of difficulty

When asked to identify their areas of difficulty, the participants suggested working with improper fractions and re-unitising to be the most problematic, though analysis of the data showed that whole number bias was also a particularly evident difficulty.

Whole Number Bias

Whole number bias (Ni and Zhou, 2005), which is the inability to view a fraction as a single quantity was evident in most students' responses. For example, see the following question where there was a considerable range of answers.

Which is the best estimate for $\frac{12}{13} + \frac{7}{8}$?

a) 1 b) 2 c) 19 d) 21?

Three out the thirteen students gave accurate and immediate answers. Four students displayed some evidence of whole number bias, they initially responded to the question by considering the numerator and denominator separately taking each as a natural number. Two initially responded with "nineteen over twenty-one" (Iris and Gill), having added each pair of numerators and denominators. One student (Betty) was very uncertain. "It must be quite big… probably 19, but I can't remember what you are supposed to do".

This was a tendency to respond to the numerator and denominator separately, taking each at face value. This relates to children's early experience of natural numbers where each number has its own unique value, which can be counted as a discrete quantity and is represented in a systematic way. Sometimes this initial understanding of number over-rides the undertaking of the fraction as a number in some children and adults (Mack in Carpenter et al., 1993). In the reflective discussions following this question, several participants considered that when faced with each number separately they would have said it was, "close to one" but when addition was included, there was an assumption in many cases that a method was required and they were uncertain as to what that might be. “My number sense deserted me,” (Jane) “it is quite obvious when you look at each number separately” This was a recurring theme with more formally laid out questions; the belief that there was an established method which should be employed but they were unable to recall it from their secondary school mathematics lessons.

Unitising and Reunitising

Unitising involves *partitioning* a whole, into *equal* parts whereas reunitising involves reconstructing the parts back to create the original whole (Lamon, 2005). Questions which required the participants to unitise and re-unitise in a flexible way, proved a particular problem for eight of the students. The focus of questions (see figure 1) was re-unitising, which involved identifying the original whole, with only one question involving unitising once the whole had been established. This focus was selected as a range of studies suggested that this remains a problem in some secondary school pupils and adults (Kieren, 1993; Lamon, 1999). This also seemed to be the case in this study, the students were generally confident with unitising but the concept of re-unitising seemed unfamiliar to the majority of the students Re-unitising was considered an important strategy whether undertaken physically or mentally and the inability to identify the base unit was considered as a factor in inhibiting development of a greater level of understanding (Mitchell, 2004). By including three questions, this developed into a learning opportunity and in most cases, a suitable strategy had been employed and the third question was approached with more confidence. This was typified by Betty, who after a less confident start tackled the question logically, using the same type of jottings she had used successfully on the earlier questions.

> **Betty:** Oh no, not another unit one! So… to make it a whole you would need seven sevenths, so nine is three sevenths. Then divide that into that, so each little bit… (circling three dots at a time) hmmm… so I need some more sevenths, so add three and… (counting up in threes). Is it twenty-one?

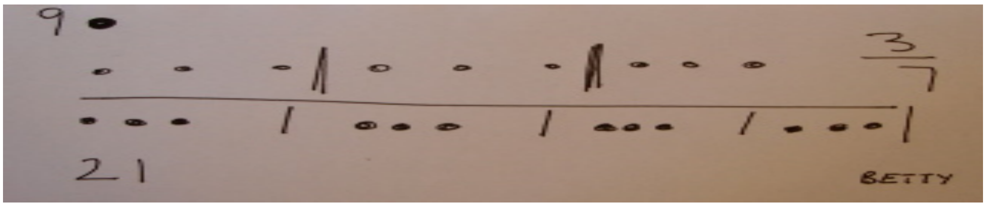

Figure 2. Betty’s jottings in response to the question: These circles represent $\frac{3}{7}$ of a unit. How many is the whole unit? ● ● ● ● ● ● ● ● ●

Working with improper fractions was considered problematic by several students who felt they lacked experience in using them and were often referred to in a different way to ‘proper’ fractions, for example as “thirty-nine over ten” or “three over two”. In general they did not appear to apply their understanding of proper fractions when working with improper fractions. In several cases they seemed to adopt a very procedural approach to aspects of the tasks and interview questions that included improper fractions. There were a range of other difficulties experienced by a smaller proportion of the participants and some very specific individual responses, which also shed light on adult understandings of fractions.

Conclusions and Professional Impact

The adoption of a phenomenographical approach proved very effective in understanding the learning of fractions from a student teacher's point of view. The students, although undergoing the same experience, brought their prior learning and attitudes to the task, which resulted in a range of qualitatively different responses. It is acknowledged that this is a very small research project based on only thirteen volunteers, however it has revealed that student teachers' knowledge and understanding of fractions is complex, individual and varied. The establishment of a supportive environment in which participants can articulate and explore their individual beliefs, understandings and misconceptions has contributed to a better understanding of the difficulties which they experience with fractions and related areas. It has highlighted several areas that have impacted on the primary mathematics initial teacher education courses in my current institution and with our work with teachers in the partnership schools. A greater level of consideration has been given to the process of reunitising and linked to the more usual focus on unitising. Another aspect, which has been identified as particularly beneficial by students, has been the inclusion of fractions when studying other areas of mathematics. The intention of this is to emphasise fractions as numbers and to reinforce their place within the number system. This has been especially effective when considering place value and measurement.

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