The case of the square root: Ambiguous treatment and pedagogical implications for prospective mathematics teachers

Cosette Crisan
Institute of Education, University of London

I report on a small-scale study rooted in the UK context that was conducted with eight volunteers from a cohort of PGCE secondary mathematics students (participants). The participants' own understanding of the square root concept and use of the associated symbol were explored and the findings revealed that they may not possess adequate subject knowledge about and for teaching this concept. Access to instructional materials, mainly textbooks and discussions with other more experienced teachers were identified as the main external sources consulted by the participants in order to refresh their knowledge of the square root concept. During this study, those participants who became aware of the shortcomings of their conceptual understanding of the square root felt, at first, uncomfortable with modifying their personal knowledge and their long held beliefs about this concept. Group discussion helped most of the participants become aware of connections between their more advanced knowledge of mathematics and the square root concept. Such awareness empowered the participants to clarify this concept for themselves and critically scrutinise the (re)sources available. A tension between employing their modified knowledge about the square root and adherence to the widely accepted view about this topic in school mathematics has also been identified.

Keywords: square root, radical symbol, Advanced Mathematics Knowledge

Background to the study

This paper originates in an informal conversation between myself and another mathematics teacher educator colleague who debated whether teachers should ask pupils the question What is the square root of 16? or What are the square roots of 16?

This conversation motivated me to carry out a review of the use of the radical symbol $\sqrt{\ }$ amongst students, undergraduates, mathematics teachers and most of the authors of school textbooks. The review reported in Crisan (2012) identified a widespread misuse of the radical symbol, as well as a lack of consistency in treating the square root concept in a large number of textbooks and other instructional materials such as GCSE and A-level examination papers together with their mark schemes, workbooks, instructional software and/or web-based content. Some materials introduced a new symbol notation $\sqrt[\pm]{\ }$, according to which the notation $\sqrt[\pm]{16}$ stands for the positive and negative square root of 16, without modelling explicitly the use of this new notation. Other materials introduced the symbol $\sqrt{\ }$, exemplifying its use such as $\sqrt{16} = 4$ to be read as 'the square root of 16, which equals 4 *or* - 4' or as 'the square root of 16, which equals 4 *and* - 4', while a handful of textbooks suggested that $\sqrt{16}$ equals 4 only.

This review highlighted the fact that there is no consistency in the way the square root concept is presented in school mathematics textbooks, not to mention the misuse of the radical symbol itself. This is worrying, as it misleads users of these textbooks such as pupils and parents/carers, but also teachers. It is known that instructional materials remain the major source used by teachers in presenting topics to their students when selecting methods of teaching (TIMSS, 1995). This raises the issue of the quality of teachers' subject knowledge that would empower them to scrutinise the authority of the (re)sources they consult and it is this issue which I will explore in this study.

Rationale of the study

Ball and Phelps (2008) argue that teachers need to be able to make judgments about the mathematical quality of instructional materials and modify them as necessary. But what knowledge is needed to make such judgments?

It is widely recognised that knowledge of the subject matter is an essential component of teachers' professional knowledge base for teaching. It includes knowledge of the subject itself, extent, depth, structure, concepts, procedures and strategies (e.g. Shulman, 1986; Grossman et al., 1989). While subject matter knowledge is necessary in teaching, there is no consensus as to what depth or breadth of knowledge is essential. There is disagreement about the extra knowledge of mathematics needed; some argue that teachers also need some additional number of years of further study of the subject at undergraduate level, while others argue that teachers need to know the curriculum but 'deeper', and so the mathematics education literature has seen a widening of the definition of subject matter knowledge over the years. More recently, Zazkis and Leikin (2009) put forward Advanced Mathematical Knowledge (AMK) as "systematic formal mathematical knowledge beyond secondary mathematics curriculum, likely acquired during undergraduate studies" (p. 2368). The authors looked at teachers' ideas of how AMK is implemented into their teaching practice. Their study called for further research to determine whether teachers' ability to identify explicit connections between AMK and mathematics taught in school is a rare gift of only a few teachers or whether specific prompting is needed to bring this ability to the surface. Related to Zazskis and Leikin's call, the focus of the study I report in this paper is to explore how an awareness of the connection between more advanced knowledge of mathematics and the square root leads to a modification of the participants' personal knowledge of this secondary school mathematics topic. Implications for their pedagogical practices are also considered.

The study

In this study the eight secondary mathematics PGCE volunteers were engaged in a number of mathematics and pedagogically specific tasks with the aim of gaining access to their conceptions (knowledge, views and beliefs) of the square root concept.

The main goals of this research were:

1. *to gain access to prospective teachers' conceptions of the square root*

by asking participants to go through a mathematics task consisting of a number of questions related to the concept of square root. It was envisaged that participants might hold competing conceptions about the square root and thus group discussion was carried out in order:

2. *to identify some of the prospective teachers' sources of conceptions about the square root* and
3. *to find out what triggers changes (if any) in their conceptions of the square root.*

Methodology

The participants

The participants were selected from a cohort of PGCE secondary mathematics students that I was teaching. The purpose of the research study was explained to the whole cohort, but the specific mathematical topic was not mentioned to the students at this stage. It was made clear to them that the study was not part of the course requirements and that it was not going to be linked with any student assessment processes. An information sheet explaining the aims of the study, what the participants were expected to do and the methods of data collection was made available to the whole cohort and confidentiality issues were discussed. The students were then invited to think about participation with this small study and interested parties were asked to email me to volunteer and to return the consent form.

The Mathematics Task

The participants were asked to take home a mathematics task, complete it and return it to me the following week, on a particular day. An excerpt of which is included in the data analysis section (Figure 1).

The task consisted of a number of questions related to the concept of square root. The aim of this task was to encourage the participants to refresh their subject knowledge and revisit some of the topics where the concept of square root comes into play. Some of the questions were designed to elicit the participants' understanding of the subtleties of the concept. The questions were designed so that they would bring to the surface the implications of the widespread misuse of the radical symbol and of the inconsistent way in which the square root concept is treated in school mathematics. At this stage, the task was situated in the mathematical space (Stylianides and Stylianides, 2010) with no pedagogical constraints, at least not explicitly at this stage.

The decision to set the task as a piece of homework and not as a test was deliberate. I wanted the participants to solve the mathematics questions using their pre-existing knowledge, at the same time, if they needed or wished to do so, being able to consult other sources such as a textbook or any other instructional materials to remind themselves of the concept (definitions, facts, examples, related mathematical topics, etc.) or, even consult their colleagues or more experienced teachers.

The group discussion

The participants were invited to work in groups of four (Group I - pseudonyms: Jan, Jemma, Jack and Joan; Group II - pseudonyms: Billy, Barry, Ben and Bea) to talk to each other about how they solved/answered the questions set in the mathematics task. The group discussions were video-taped, while at the same time I took written notes of some of their explanations and comments. I probed further any issues that arose during the group discussions.

As the participants discussed their solutions and answers to the mathematics task questions, the need to clarify/defend/justify a definition of the square root of a positive real number surfaced. During the discussion, implications for teaching about square roots arose naturally, either through the participants' reflection on how they were taught the topic or on how they would teach the topic themselves. Indeed, Stylianides and Stylianides (2010) argue that when working with prospective teachers the answers to the questions posed in the mathematics tasks cannot be sought in a purely mathematical space, but rather in a space that intertwines content and pedagogy. Immersion of the participants' mathematical work in the pedagogical space was taken further through a further task using fictional pupils' scenarios.

The fictional pupils' scenarios

The participants were asked to give written feedback to three fictional pupils' responses (Emma-KS3, Peter-KS4 and Lucy-KS5) characterised by a subtle mathematical error to a question involving the square root, throwing further light on the choices the participants made about treating this concept. This is a well known approach proposed by researchers such as Biza, Nardi and Zachariades (2007) to be employed in a teacher education context "as tools for the identification and exploration of mathematically, didactically and pedagogically specific issues regarding teacher knowledge" (p. 308).

The intention with this task was to encourage the participants to reflect further on their own conceptions of the concept of square roots in the light of having done the mathematics task and having discussed the mathematics tasks questions as a group. Excerpts of the fictional pupils' scenarios are incorporated in the data analysis section (Figures 2 and 3).

Data analysis

I present the participants' approaches to solving some of the mathematics task questions, supporting their written and oral explanations with data collected during the group discussions and some of their written feedback to the fictional pupils' scenarios.

The participants' conceptions of the square root

Overall, the participants provided a variety of answers to the mathematics task questions.

Q1. Find the square roots of the following numbers. Write the answers in the boxes provided and where necessary, show your workings out.

36 , 225 , 0.64

Q2. Answer the following questions:

$i)\sqrt{4}=?\ \ ii)\sqrt{89-25}=?\ \ iii)\sqrt{9^2}=?\ \ iv)\sqrt{a^2}=?\ \ v)\sqrt{25y^2}=?\ \ vi)\sqrt{(8c+1)^2}=?$

Q3. Solve the following equations to find x. Explain how you work it out.

$i)\,x^2=16,\ ii)\,x^2=a^2$

Figure 1. (Part of) The Mathematics Task

All the participants were happy with their answers to Q1 ($\pm 6, \pm 5, \pm 0.8$) and Q2 i) ± 2 and ii) ± 8. "I'll always put $\pm$ because I am so used to it, but I did get in a muddle with some of questions in the HW," explained Jack and the others agreed with him. They all encountered some difficulties with the rest of Q2. The following answers and alternative explanations to Q2 iii) $\sqrt{9^2}=?$ were provided by both groups:

1) $\sqrt{9^2} = \sqrt{81} = \pm 9$, 2) $\sqrt{9^2} = (9^2)^{\frac{1}{2}} = (81)^{\frac{1}{2}} = \sqrt{81} = \pm 9$,3) $\sqrt{9^2} = (9^2)^{\frac{1}{2}} = 9^{2\times\frac{1}{2}} = 9^1 = 9$ and 4) $\sqrt{9^2} = 9$ (as "the square and square root cancel each other"). All four explanations were regarded as being valid and the participants did not seem to be able to find a 'fault' in their reasoning, which seems to contradict the obvious equality $\sqrt{9^2} = \sqrt{81}$. "This is how we were taught since very little" (i.e., $\sqrt{81} = \pm 9$), said Jan and so when encountering disagreements or ambiguities in their solutions, the participants worked on the premise that their knowledge is correct, hence looking elsewhere for resolving the issue.

The discussion moved on the Q3 *i*) solving $x^2 = 16$. All the participants were in agreement that the solutions were $x = \pm 4$. The solutions were reached either by solving the equation by factorisation or by using the graphical approach or by 'taking the square root' of both sides. Jan explained that taking the square roots of both sides of the equation gives $\sqrt{x^2} = \sqrt{16}$, hence $x = \pm 4$ since $\sqrt{16}$ equals ± 4 and the rest of her group seemed happy with this explanation. The same approach to finding the solution of the same equation was put forward by Billy in Group II. However, he changed his mind very soon after offering his explanation,

> Actually, strictly speaking that is not right, is it? Looking at it now, I would amend it to say that $x = \pm\sqrt{16}$ since $\sqrt{x^2} = \pm x$ and $\sqrt{16}$ equals 4.

Nobody responded to Billy's comment, who himself did not look convinced at what he had just said.

The next question Q3 *ii)* in the mathematics task asked for the solution of $x^2 = a^2$. The use of the parameter *a* instead of a specific real number on the right hand side of the equals sign made things slightly more problematic. For example, the participants in Group I were not sure on which side of the equals sign to include the ± sign after taking the square root of both sides. While all the participants seemed content with $\sqrt{a^2} = \pm a$, some doubts were raised by Jack (Group I) and Billy (Group II) about whether $\sqrt{x^2}$ should also equal $\pm x$. In the end, however one wanted 'to look at' the square root, the matter was quickly settled when the participants realised that either $\pm x = \pm a$ or $x = \pm a$ yields the same solutions to the equation.

The participants in Group II had a similar debate when comparing each other's answers to Q2*v)* asking them to simplify $\sqrt{25y^2}$. The participants soon realised that they did not need to resolve their disagreement about whether $\sqrt{y^2} = y$ or $\sqrt{y^2} = \pm y$, as multiplication by $\sqrt{25} = \pm 5$ gave the same answer, namely $\pm 5y$.

Almost one hour into their group discussion, the participants started to become less concerned with agreement over the answers and more interested with the square root concept itself and settling the ± issue.

External sources of conviction

Most of the participants' sources of conviction, which they used in order to justify their answers, were external in nature. The participants relied on what they remembered from school or what they learned from the instructional materials they brought along to the group discussion. The participants became aware of the inconsistencies of how the square root was presented in textbooks. "They [authors of textbooks] don't care. I'm disappointed about this lack of agreement," said Billy, while Ben said, "I'd like to go to the National Curriculum exam board because I would feel secure if I knew what people will go for, who makes the decisions? Who wears the hats?" On a frustrated note, Jack summarised his Group II's desire, "We want an answer!"

Internal sources of conviction

The lack of agreement between the resources browsed led to an interesting turn in the discussion. Group I set out to bring clarity to the concept. For example, Jan talked about the possibility of the more mathematically rigorous origin of the square root as the output of the inverse of the square function. Drawing on her first year undergraduate analysis course she concluded that only the positive value should be accepted as a correct answer and explained this to the whole group in great detail. While agreeing with Jan, Jemma, who "only studied an applied mathematics course as an undergraduate", enquired about why the square root needs to be a function, and as such have only one output given the input; she was not clear about why one should not consider the square root as just a relationship, or a mapping. Jan concluded, "It seems to me that as a function and inverse, we can only accept one answer, while as a process it is acceptable to have two answers." After some discussion, while Joan pointed out to the $\sqrt[\pm]{81}$ notation she came across in a textbook, Jan said, not very convincingly, that maybe the symbol usage needs to be addressed, namely that the radical symbol should perhaps only be used for the positive square root of a number. Her point was acknowledged by the whole group, who sighed with relief for finally reaching a conclusion.

In Group II, the participants also expressed their frustration with the polemic surrounded the + or - . Bea remembered that she was taught at college that $\sqrt{x^2} = |x|$ for any x real number, and so $\sqrt{a}$ can only be a positive real number. She went on to explain how based on this fact $\sqrt{16} = 4$ and that taking the square root of both sides would then yield $|x| = \sqrt{16}$, hence $\pm x = 4$ resulting in $x = \pm 4$. This clarified the presence of $\pm$ for the other participants, some of whom needed help to remind themselves about the modulus function and why the relationship $\sqrt{x^2} = |x|$ holds true in the first place. The participants in Bea's group realised that using this definition of the square root the ambiguities encountered before would be eliminated and so they happily adopted it for solving the other mathematics task questions.

Pedagogical decisions

The participants were asked to take home the fictional pupils' scenarios.

Give the answer to the following questions:

a) $\sqrt{4} = ?$ b) $\sqrt{36} - \sqrt{9} = ?$ c) $\sqrt{7^2} = ?$ d) $\sqrt{25-9} = ?$

Emma responded as follows:

a) $\sqrt{4} = 2$ *and* -2 b) $\sqrt{36} - \sqrt{9} = \pm 6 - \pm 3 = 3, 9,$ *got stuck here, sorry...*

c) $\sqrt{7^2} = 7$ *because square root and square cancel each other*

d) $\sqrt{25-9} = \pm 4$

What comments would you make to this pupil with regard to her answers?

Figure 2: Emma's (Year 8 - KS3) scenario

Jan, in Group I, who so eloquently talked about the ambiguity of the radical symbol notation, concludes her feedback to Emma, the KS3 fictional pupil that " $\sqrt{7^2} = \sqrt{49} = \pm 7$ so when you see $\sqrt{\ }$ you must consider both the positive and the negative roots", a position in contrast with the one she reached during the group discussion. However, in her feedback to Lucy's (KS5 fictional pupil) solution to the same question, she explains that "$\sqrt{7^2}$ can only equal to 7, as the square and the square root cancel out each other".

Bea in Group II was consistent in her feedback that the radical symbol is to be used for positive answers only, as $\sqrt{x^2} = |x|$. Ben, too employed the modulus, providing a detailed feedback to Lucy's solution seen below.

Solve the following equation: $3x^2 = a$

Lucy responded as follows:

$3x^2 = a$ *divide both sides by 3*

$x^2 = \frac{a}{3}$, *square both sides and so* $\sqrt{x^2} = \sqrt{\frac{a}{3}}$, $x = \pm\sqrt{\frac{a}{3}}$ *is the solution*

What comments would you make to this pupil with regard to her answers?

Figure 3: Lucy's (Year 13 - KS5) scenario

He points out in his feedback the missing intermediate steps in Lucy's answer, "The statement $\sqrt{x^2} = \sqrt{\frac{a}{3}}$ so $x = \pm\sqrt{\frac{a}{3}}$ is fine as long as the intermediate step is $|x| = \sqrt{\frac{a}{3}}$ ".

While the whole group employed this definition when attempting the other mathematics tasks questions, from a pedagogical point of view, Billy did not think this definition would be of much use since the square root is introduced much earlier than the concept of modulus in secondary school education.

Barry in Group II writes that the radical notation is assigned to the positive square root by convention, "However, by convention, we usually take $\sqrt{4}$ to just mean the positive root, i.e. 2.*",* showing that he, too adhered to Bea's unambiguous way of working the square root.

Concluding remarks

The tasks in which the participants were involved in this small study provided a context in which their knowledge, preferences and choices were brought to the surface.

The mathematics tasks created some instability in what the participants knew about the square root and the discussion of how they tried to resolve the discrepancies brought out into the open their knowledge and interpretations of the subject matter and enabled the participants to question further or clarify these concepts for themselves when needed. The activities carried out during this study offered the prospective teachers opportunities to engage with the subject matter at a level deeper than simply recalling their existing knowledge. The tasks were triggers for reflection and introspection, "I thought I knew all about square roots until I worked on this homework." (Jemma) and during discussions other mathematics concepts came under consideration.

This study adds to Zazkis and Leikin's (2010) call for a more articulated relationship between AMK and mathematical knowledge for teaching. The participants in this study benefitted from recalling some of their AMK (function, relation, mapping, inverse function, modulus function) which they studied more formally and in depth in their undergraduate studies. A few of the participants (Ben, Jan and Bea) became explicitly aware of the connections between their AMK and the square root concept under scrutiny. Their articulation of such connections triggered recall of these advanced topics and the implications for the other participants' square root definition. All participants realised that with the new understanding gained about the square root, the ambiguities initially encountered were removed and so they were able to answer the rest of the questions in the mathematics task without further confusion or even disagreements.

However, their feedback to pupils' scenarios highlighted the tension between the widely accepted school definition of the square root, together with the use of the radical symbol and their own modified conceptions. Five of the eight participants in this study decided in the end to adhere to the widely accepted school practice of treating the square root and misuse of the radical symbol, i.e. $\sqrt{16} = \pm 4$.

The findings of this very small scale study suggest that more needs to be done to empower prospective teachers not only to scrutinise (re)sources through applying their AMK to ideas in the secondary school mathematics curriculum, but also to challenge the accepted, ambiguous treatment of concepts and ideas in school mathematics.

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