How-many-ness and rank order—towards the deconstruction of 'natural number'

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This paper concerns a philosophical, but highly practical, issue arising at the interface between mathematics and education, and I claim that mathematics education offers insights where mathematical philosophy has ground to a halt. More specifically it concerns the two related but distinct concepts of how-many-ness (alias *quotity*) and rank order, whose separate identities are traditionally obscured by the language of 'number'. (Sometimes they are called 'cardinal number' and 'ordinal number'. More often they are wrapped up together as 'natural number'.) The teacher of young children has the advantage over the philosopher that she works with people before they have acquired all the prejudices of their native language, and we shall build on the analysis of *counting* by Gelman and Gallistel (1978), concluding that 'natural number' as normally conceived is something of an illusion, for only quotity has the properties expected of 'number', while rank order is a mere *quality*.

Keywords: natural number, cardinal/ordinal, quotity

Introduction 1: the mystery of 'natural number'

Nothing does more to unite the mathematician with the person in the street than their shared belief in 'number'. Yet few things have proved more elusive under close investigation or more difficult to pin down in a definition.

Mathematicians speak on the issue with many voices. Dedekind declared enthusiastically:

> Of all the aids which the human mind has yet created to simplify its life… none is so momentous… as the concept of number. (Ewalt, 1996: 837)

In this, however, he raises eyebrows at the claim of human authorship, for Plato had assigned all mathematical objects to a domain of external objective super-reality. Kronecker struck a famous compromise:

> The dear God has made the whole numbers, all else is man's work. (Grattan-Guinness, 2000: 122)

But this raises the enormous issue of the higher 'types of number'. Aristotle, with a wisdom that may yet return to favour, divided mathematics into the study of multitude (*plethos*), which is discrete, and magnitude (*megathon*), which is continuous, and this distinction was still retained at the time of Newton, who identified continuous number (real number) with the ratio of continuous quantities. But in the development know as the *arithmetisation of analysis* it came to be believed that the study of continuous quantities could be reduced to that of 'real number', which in turn could be defined in terms of other numbers and ultimately natural number. Whereas Aristotle made mathematics stand on two legs, the modern mathematician believes it can balance on one. To challenge this thesis is to undo the work of the last two hundred years and clearly requires more space than a single article. I will therefore say no more about it here. But it does bear on our immediate issue to this extent, that the modern number system requires a foundation, and that foundation is looked for in 'natural number'.

Alas 'natural number' has proved to be less straightforward than expected. Peirce put his finger on one critical issue when he asked

> whether the cardinal or the ordinal numbers are the pure and primitive mathematical numbers (Peirce, 1933: para. 658).

In other words does the essence of natural number lie in *how-many-ness* or in *rank order*? Broadly speaking, Dedekind, Peano and Peirce himself (after an initial flirtation with cardinality) favoured rank order ('ordinal number'), while Frege, Husserl and Russell went for how-many-ness ('cardinal number'). Only Cantor provides a more balanced view, suggesting that cardinal number (which he sometimes called the 'power' of a set) and ordinal number (alias 'ordinal type') might be two independent, if related, concepts. But he applied this analysis only to transfinite numbers, supposing that ordinary numbers somehow contain the potentiality for both concepts within a single entity. (See Cantor, 1900)

One modern view—sometimes promulgated even to teachers—is that natural numbers are defined by Peano's axioms. Yet, as Russell at least realised at the time, these axioms are satisfied by any rank-ordered sequence (in Russell's terms any *progression*), and from the available candidates he wanted to select "such as can be used for counting common objects" (Russell, 1920: 12). And there is now serious doubt as to whether the axiomatic method is suitable at all for pinning down a specific entity like natural number—see Wilder (1967).

The whole situation is admirably summarised by Goodstein:

> It is surely a very remarkable thing that despite the range, power and success of modern mathematics, the concept of natural number, on which the whole edifice rests, is still something of a mystery. (Goodstein, 1965: 68)

But in this summary Goodstein unwittingly perpetuates the two implicit assumptions that render the issue so intractable: the assumption that 'the whole edifice' must have a single foundation, and the assumption that that foundation is provided by 'natural number'.

Introduction 2: towards a solution and the role of mathematics education

The way to a solution, I believe, is sign-posted by three insights, all of them more than half a century old. The first is from Frege

> A frequent fault of mathematicians is their mistaking symbols for the objects of their investigation (Frege and Gabriel, 1984: 229)

Although Frege was perhaps thinking of higher mathematics, his stricture applies very well to arithmetic. In geometry, when the vertices of a triangle are labelled *A, B, C*, no-one would for a moment think that the 'objects of investigation' were letters of the alphabet, because the idea of a geometrical point is well enough understood and the letters are mere labels. But in arithmetic, where the labels are the numerals '1', '2', '3', it is far from clear what objects are being labelled, so that one is tempted to cling to the labels as if they were the real thing.

The way to track down the elusive 'objects of investigation' in such cases is indicated in the second insight, from Wittgenstein, who notes

> … the question, 'What do we actually use this word or this proposition for?' repeatedly leads to valuable insights. (Wittgenstein, 1961: 65)

Again Wittgenstein was presumably thinking more generally—and we need not here concern ourselves with the further views on language that he developed later—but what he says applies with special relevance to arithmetic, covering specialist symbols like numerals as well as number-words. There is a corollary to this insight: if it turns out that a symbol is used in two or more ways, then it has two or more meanings.

And this leads to the third insight, from Ayer, who refers to

> the superstition that to every name a single real entity must correspond (Ayer, 1946: 42)

If it turns out that a name (like, for example, 'natural number') refers to two or more distinct concepts (as it might be, say, how-many-ness and rank order), then it is erroneous to suppose that there exists some third thing (perhaps called natural number) over and above the concepts already identified, which somehow subsumes these into a whole. To this also we may add a corollary: once it is established that a word or other symbol is used to denote two or more separate concepts, then it may be helpful, at least in serious analysis, to given the concepts proper names of their own (as, for example, 'how-many-ness' and 'rank order' or perhaps 'cardinality' and 'ordinality') and drop the original name altogether because it is merely a source of confusion. For example, if one wishes to delineate the relation between how-many-ness and rank order, the task is made unnecessarily difficult if the two things are both called 'number'. (Indeed this common name may disguise the fact that there is any relationship to be investigated anyway.)

It is not perhaps obvious that mathematics education has any special contribution to make to the resolution of these fundamental issues. Teachers employ the vocabulary of number as freely as anybody. Nor are teachers especially scrupulous in distinguishing number-symbols from the things they denote. (And in any case such delicacy would be unhelpful if the things denoted were simply called 'numbers' without any finer discrimination.) In consequence the literature of mathematics education, like that of mathematical philosophy, contains much that is open to criticism in the light of the Frege/Wittgenstein/Ayer insights.

I believe nonetheless that teachers of young children (and those who teach them, advise them, watch them and do research in the classroom) do enjoy some advantages when approaching the foundations of mathematics. Children (some of them all the time and all of them some of the time) find mathematics difficult, and work has to be done to analyse the sources of their difficulties. And those who work with young children meet some of their fellow human beings before they have acquired all the prejudices of their native language. Anyway, for whatever reasons, the literature of mathematics education contains insights into mathematics as well as education, and we shall draw on some of these in the present paper.

Counting

Although mathematicians and philosophers often refer to counting as if it were the bedrock of arithmetic (if not of all mathematics), they also seem to regard it as too elementary and familiar to require further analysis. That task has therefore been left to mathematics education.

Counting in the simplest sense is the mere recitation of the number-names in order, but the counting *of objects* requires a one-to-one coordination between the spoken words and the objects in a given set. It thus provides a prime example of the process of *symbolisation* or *throwing together*. It is for this reason, of course, that number-words and numerals are often called 'symbols'. However, that term could with some justice be applied to the things that are thrown from the other side, and in their masterly analysis Gelman and Gallistel (1978) achieve a little extra clarity by avoiding the actual word 'symbol' and calling the number-words simply *tags*.

These authors observe that counting is governed by several principles. The tags themselves must have a fixed order (the *stable-order principle*). The objects being counted need not have any pre-existing order, but, even if they do, you can ignore it and count them in any order you like, so long as each object gets a unique tag (the *one-to-one principle*). Whatever the order of the counting, the last-used tag will always be the same (the *order-irrelevance principle*). Finally, the last-used tag must be re-interpreted as saying how many objects in the set (the *cardinality principle*).

By far the weightiest of these principles is the last. The order-irrelevance principle has its own importance because, whereas the tags are at first attached to individual objects, the last-used tag, being invariant, can now be thrown also against the complete set-of-objects. But the cardinality principle goes further, because that tag is not just a label for the set-of-objects, but states *how many* objects are in the set. It is a label for the *degree of how-many-ness* of the set, or (in one sense of the term) the *size* of the set, as witnessed by the fact that the same tag would be attached to any other set of the same size. In order to use the tag in this new sense a person must already have some notion of how-many-ness or size-of-set, and it is important to realise that, whereas the process of counting provides a *technique* for finding how many, the *concept* of how-many-ness has its origins elsewhere.

Approaching the matter from the direction of cognitive development, Schaeffer, Eggleston and Scott (1984) note that the degree of how-many-ness of a small set (say, up to five elements) can be seen at a glance without the need for counting, and they suggest that, if counting is now applied to such a set, the child may learn that, in these cases at least, counting gives the same answer. From there she will—according to the *skills-integration model* they employ—go on to make the required generalisation.

Approaching from a different perspective, it may be argued that the process to which how-many-ness is most intimately related is not counting at all but the direct matching of sets-of-objects by one-to-one correspondence. When this is done, it will be found that in most cases one set is larger than the other (contains more objects), but sometimes they are the same size (in Frege's term *equi-numerous*). This equivalence relation of equi-numerosity underlies the abstract concept of how-many-ness, and the order relation in which it is embedded (has-more-than) underlies the abstract order relation on degrees of how-many-ness (three-ness is greater than two-ness).

Whatever the precise analysis, however, the conclusion must be that the use of counting to find the degree of how-many-ness of a set of objects requires a grasp of how-many-ness as well as a grasp of counting, otherwise the purpose of the exercise will be missing. That is the essence of the cardinality principle.

We may note that, when counting is mentioned in academic mathematics or mathematical philosophy, there seems to be no appreciation of the cardinality principle at all. Thus Benacerraf (1965: 50) distinguishes *intransitive* counting (the mere recitation of number-names) from *transitive* counting (the counting of objects), but the latter he takes to be all the rest of the process, making light of the difference between throwing a tag at an object and throwing it at the whole set, let alone throwing it at the degree of how-many-ness of the set. Peirce (1933: para 156) identifies what he calls The Fundamental Theorem of Arithmetic, but this title he bestows on the order-irrelevance principle, not the cardinality principle. He seems to think it sufficient that a numerical tag should be associated with a (finite) set, and ignores the further step of associating it with an *attribute* of the set.

Rank order and the use of letters as numerals

Although the size of a set is unaffected by the order in which the elements are counted, the process of counting does nevertheless have the incidental effect of placing the objects in order, at least temporarily. But, of course, some sets of objects—for example the pages of a book or the houses in a street—are ordered permanently, and here numerical tags can be used permanently to indicate position in the sequence.

It is worth noting, however, that alternative notations are available for this ordinal purpose. There are, firstly, the dedicated adjectives, 'first', 'second', 'third' etc. Whether a street is called 'Fifth Avenue' or 'Avenue Five', whether a house is called 'Number 6' or 'Sixth House', whether a person is called 'Second-in-Command' or 'Number Two'—these are matters of style not mathematics, for in either case the concept being indicated is rank order. Only the symbols are different.

Another alternative is provided by letters of the alphabet. The sections of a report can be labelled Section *A*, Section *B*, Section *C* just as well as Section 1, Section 2, Section 3 (or Section I, Section II, Section III). Letters are suitable for this purpose because, like numerals, they themselves possess rank order. They also provide, like numerals, a comprehensive system because, after the first twenty-six places, they can be combined into compound tags by the principle of place value. (After *Z* you take *AA*, *AB* etc, and after *AZ* you take *BA*.) In theory letters of the alphabet could replace numerals in all ordinal applications. For example, the pages of a book could be lettered instead of numbered. I think that the use of letters has a special role in conceptual analysis, where it can avoid the ambiguity of using numerals simultaneously in both an ordinal and a cardinal role.

Note in passing that in this alphabetic system there is no sign corresponding to zero. It is sometimes said that the principle of place value depends on the introduction of zero. But this is an over-simplification. Zero would not be needed with numerals either if these were used only for rank order. (It would then be acceptable for 9 to be followed by 11, 19 by 21, and 99 by 111.) Zero is only needed for the denoting of how-many-ness.

Addition

The degrees of how-many-ness—which we are calling 'one-ness, 'two-ness' etc—possess a rich structure. In the first place, they form a rank-ordered sequence. Normally one-ness is the first, two-ness the second and so on. If you start with zero-ness, then zero-ness is the first and one-ness the second. But either way they possess *rank order*. For this reason they can be represented by the points of a (discrete) number-line—see further below.

The degrees of how-many-ness also possess the normal operations of arithmetic, starting with *addition*. In order to add four-ness and five-ness, for example, you take any set of four objects, combine it with any (disjoint) set of five objects, and find how many objects in the union. The mathematician will note that this operation is *well-defined* because the result is independent of the choice of objects. The teacher will note that the usual symbolic statement '4+5=9' can be interpreted either at the concrete level, meaning that a set of 4 objects combined with a set of 5 objects gives a set of 9 objects, or at the abstract level, meaning that the sum of four-ness and five-ness is nine-ness. In any matter to do with addition of degrees of how-many-ness, however abstract the setting, if one gets stuck or fails to remember a result, one can always appeal to sets of concrete objects.

At this point it is necessary to stress a very negative fact: **rank-ordered sequences in general do not possess an operation of addition**. Up to a point this is obvious. You do not 'add' the fourth person in a queue to the sixth person to get the tenth person. Nor is there any sense in which pages 7 and 8 of a book can be combined to give page 15. Unfortunately neither mathematicians nor teachers have fully faced up to this natural deficiency, and they have tried to make up for it in various ways, particularly when they have in mind the sequence of 'numbers'.

One way, popular in mathematics, is to *impose* a kind of addition on rank-ordered sequences. Take, for example, the letters of the alphabet—and you can take them either as objects in their own right or as tags for other objects. You can then decide that $A+A$ will be, say, B (or any other letter), and complete an 'addition' table in the obvious way to get $A+B=C$, $A+C=D$ and so on, making use of the successor relation. This is often called 'definition by induction'. I think a better name is *definition by imposition*. The trouble is, of course, that the result does not correspond to any natural feature of the domain on which it is imposed. In this way you can 'add' the kings and queens of England so that, for example, Edward VI, the successor of Henry VIII, becomes the 'sum' of Henry VIII and William the Conqueror!

Another way of bringing in addition, popular in teaching, is to involve *displacements* or jumps along a sequence: a jump of 3 steps, combined with a jump of 4 steps in the same direction, gives a jump of 7 steps. This is a genuine addition, being an example of the *vector addition* that can be found in spaces of any dimension, both discrete and continuous. It may be remarked, however, that, if one were setting out to investigate jumps along a dotted line, and anticipated that the sizes of the jumps would be denoted by numerals, it would be asking for trouble to use numerals to tag the points of the line at the same time—letters would be better. (Any alternative would be better.) But this is what happens in the primary classroom. And trouble does ensue. If, using letters, you start at point C and move 4 steps along, you will finish at point G, an operation that might with a good notation be symbolised as $C \xrightarrow{+4} G$. However, if the points (as well as the displacements) are labelled with numerals, the formula that springs to mind is $3+4=7$, and this gives a kind of hybrid addition or *pseudo-addition* in which the first and last numerals are tags for points and the '4' says how many steps in the jump.

The attempt can be made to justify pseudo-addition in terms of vector addition, but this is more tricky than it looks, partly because it means abandoning the original assumption that some of the numerals are simply tags for points, and partly because it requires the introduction of a zero-point or origin. One can also try to justify pseudo-addition by counting suitable subsets of points, but this raises doubts as to whether it is points or steps that are to be counted.

Fuson (1984: 219), taking a purist view, argues that the number line is a 'measure model', not a 'count model', and insists that numbers are represented by lengths and not by points at all. This is very close to an endorsement of what I am calling vector addition, and certainly rules out pseudo-addition. But I have one serious quarrel with this analysis. In my view the number line is not a 'representation of number' at all because 'number' does not exist.

Among the things that do exist are degrees of how-many-ness, numerical tags, and various geometrical models including the *continuous half-line* and the *dotted half-line* or *stepping stone model* (both of which are sometimes called number lines). The stepping stone model may certainly be used to represent the degrees of how-many-ness, demonstrating their rank order, so long as it is realised that in this model there is no representation of addition. (The continuous half-line can similarly be used to represent degrees of any continuous quantity, with the same proviso regarding addition.) But the geometrical models may also be studied for their own sakes, and in the study of jumps along a dotted line number-symbols will naturally be used to indicate how many steps in a jump. Note, however, that the objects here are steps, jumps, the sizes of jumps, and numerical tags. It only confuses the situation to talk about 'numbers' as well.

Conclusions

In our analysis the cardinality principle is important because it brings together counting and how-many-ness (the ordinal and the cardinal). The mathematician's lack of interest in it can perhaps be blamed on the division of that community into two camps, neither looking for rapprochement because each believes that its own position provides a complete account of the entire subject. In mathematics education such a reductionist stance is less common. It is true that Padberg and Benz (2011) detect "two very different tendencies" (*zwei sehr unterschiedliche Ansätze*), one associated with Piaget, stressing things like invariance and one-to-one correspondence, and favouring the cardinal, and the other, based on the skills-integration model, laying more stress on counting, and favouring the ordinal. But it is hard to believe that practising teachers have ever entirely neglected either how-many-ness or rank order, and Padberg and Benz themselves propose that primary teaching should be based on the concurrent development of several different "aspects of number", including a cardinal aspect and an ordinal aspect.

This plan is broadly attractive on both practical and theoretical grounds. My objection is that, while it follows the Wittgenstein insight by looking for usage, it is less attentive to the insight of Ayer on the avoidance of spurious objects. In fact the very phrase 'aspects of number' implies that the various concepts identified by studying usage, notably how-many-ness and rank order, are merely 'aspects' of some further thing called 'number'. This is Ayer's superstition without apology or mitigation. It can be avoided if one says, more simply, that

> Children should study the elementary uses of number-symbols, especially in the denoting of how-many-ness and rank order.

There is no need for the word 'number' at all.

I am not the only person to deny the existence of 'natural number'. Benacerraf (1965: 73) also reached the conclusion that "there are no such things as numbers". Benacerraf, however, was too pessimistic. He was looking for number in rank ordered sequences generally and was concerned (rightly) at their lack of uniqueness. He gave no credence to *how-many-ness*, which is the hero of my own analysis, and in fact possesses all the properties that are normally looked for in 'natural number'.

What's in a name? you may ask, and that is a good question. I have tried to argue that the term 'natural number' is too soiled by misuse to merit any continuing place in serious analysis. It also carries overtones of the arithmetisation of analysis—and that is seriously prejudicial to further developments. However, 'degree of how-many-ness', which I have used in the present paper is cumbersome and inelegant. It could perhaps be replaced by 'cardinality', which emphasises the contrast with rank order. For further work, however, my preferred term will be 'quotity' from the Latin *quot*, meaning how many. This term has been introduced into mathematics by Lucas

(2000: 98), who points out the contrast with 'quantity' from *quantum*, meaning how much. The parallels and contrasts between quantity and quotity provide a further rich field of enquiry. (And both can be compared with mere *qualities* such as shape and rank order, which lack addition.) But, of course, there is still much to be done to establish the credentials of quotity itself.

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