Progression towards understanding functions: What does spatial generalisation contribute?

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We focus on a 'typical' task in which students have to give a functional generalisation in algebraic form of a growing sequence of spatial structures. We analyse the contribution of this task to a coherent knowledge of functions. Despite a plethora of research about misconceptions and the teaching of functions, little is known about the overall growth of students' understanding of functions throughout schooling. We aim to map the development of students' understanding of concepts which contribute to understanding functions in two different curriculum systems: the UK and Israel. The research uses a survey instrument that was developed in collaboration with a group of teachers and the task for this paper is one of six that span several routes to understanding functions. Our data appears to contradict some other studies as well as to suggest conjectures about how students' willingness to use covariational reasoning depending to some extent on task features.

Introduction

The function concept is both an explicit and implicit foundation for advanced study in mathematics itself and as a tool in other subjects. The roots of function understanding do not consist of a single hierarchical pathway (Schwindgendorf, Hawks and Beineke, 1992). This paper examines one small part of a project to construct a description of progression towards functions based on probing students' understanding. The research has several stages and we are currently analysing the implementation of a survey instrument that is being used across two countries.

Learning does not only depend on the written curriculum, it also depends on school and classroom context, teaching, and possibly on the level to which teachers are 'functions aware' (Watson and Harel, 2013) and on national expectations through assessment regimes. In order to juxtapose such national expectations we are working in two countries: UK and Israel. The curriculum in the UK has an informal approach to functions, not requiring a formal treatment until year 12[2] for those who continue to advanced study, but younger students will, for example, generalise sequences and meet input-output models as 'function machines'. In the Israeli curriculum approaches to functions are more explicit for younger students and the word and the notation are introduced in year 7. All project teachers are 'functions aware' due to their self-identified levels of mathematical knowledge. In this paper we outline our research approach and demonstrate its application to one task (out of six) in one national context (UK). Due to issues of gaining ethical approval the data collection from Israeli schools will take place in 2014.

We developed a survey instrument over several design cycles working closely with teachers to adapt existing tasks and develop new ones (Wilmot et al., 2011; Swan 1980). We then selected an optimal set of questions that addressed distinct routes to understanding functions that we had identified from the literature. The questions had

[2] We are using UK years in this paper.

to be accessible for students in years 7 to 13, and had to be completed in one lesson. This survey was implemented in two schools in the UK to provide data for analysis to learn about progression towards functions in secondary school, while being aware of grouping, teaching, curriculum, prior attainment, and other variables. The schools and teachers were similar in many ways (size, socio economic factors, ethnicity, stability, qualifications) but differences in spread of prior attainment in constructing teaching groups are likely to have an impact on learning.

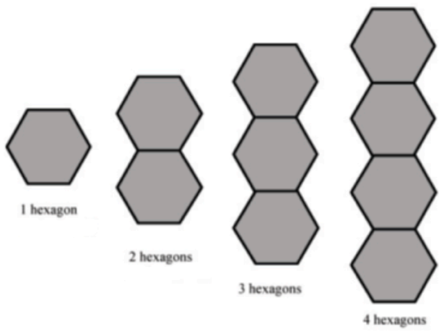
Spatial sequence generalisation tasks

The findings we present in this paper are from a spatial growth pattern generalisation task. There is a considerable body of literature about such tasks, investigating either the processes of generalisation or the effects on algebraic understanding more generally (e.g., Carraher, Martinez and Schliemann, 2008; Dörfler, 1991; Radford, 2006, 2008; Stacey, 1989). Many studies have been conducted to identify processes in building generalisations from spatial sequences. These studies vary in the types of patterns, the population studied and their perspectives and accordingly in the categories of generalisations they present. For example, Dörfler (1991) defined two types of generalisations: empirical, referring to the recognition of common features, and theoretical 'systems of action', identifying variables and expressing prototypical relations between objects. Radford (2006) identified three generalisation strategies used with patterns: algebraic generalisation: 'grasping' a commonality, generalising for all terms and forming a rule; arithmetic generalisation; and inferences based on local guesses. Rivera and Becker (2008) distinguish between constructive and deconstructive forms of generalisation. The constructive form results from perceiving figures as consisting of non-overlapping parts, exhibiting the standard linear form $y = mx + b$. The deconstructive form is based on initially seeing overlapping sub-configurations in the structure and would lead to separately counting each sub-configuration and taking away parts (sides or vertices) that overlap. Constructive generalisation seemed easier for middle school children to establish while deconstructive was more difficult to achieve. Stacey (1989) defined four main generalisation approaches: recursive; counting from drawing; searching for a functional relationship from a figure; and making an incorrect proportional assumption, using the ratio $f(x) = nx$, when the relation is $f(x) = ax + b$ $(b = 0)$. Most studies agree that students find it difficult to reach theoretical generalisations and that they tend to begin with a recursive approach to sequential data. In England recursion is often referred to as 'term-to-term' and a functional relationship as 'position-to-term', where 'position' refers to sequential position. Several researchers claim that presentation influences approaches since presenting data in order can encourage a term-to-term approach (e.g. Stacey, 1989; Steele, 2008).

The aim of these studies was to focus on obstacles to generalising the underlying function. It cannot be claimed, however, that students who succeed have any sense of functions, even though some authors describe this as a 'functional approach'. As Dörfler points out (2008), these tasks only model particular kinds of function, those that are expressible as strings of arithmetic operations that relate to continuing patterns with integer inputs. To have a sense of 'function' would require variation and comparison of functions and their properties (Carraher et al., 2008).

Our aim was not the same as these studies. We used a spatial sequence generalisation task as one of six tasks, all of which address components of the function concept. We expected this task to provide information about how students try to identify relations when prompted to state different kinds of generalisation. By seeing what they use and how they try to generalise, whether successful or not, we can identify ways of attending to data and spatial information that might form a basis for future knowledge of functions. Spatial tasks can provide: early experience in modelling relations between two variables; opportunities to explore co-variation, relating to gradients and early calculus; experience in analysing simple functions in given domains; and experience in expressing input-output relations algebraically. Our analysis, therefore, does not imply preference for any approach. Dörfler (2008: 147) says that different approaches "... shed different light on the common underlying functional relationship." For example, term-to-term perception is a plausible pre-concept towards gradient if variation in the output variable is related to variation in the input variable to show some sense of co-variation (Carlson et al., 2002). As well as relating to gradient, co-variation fits well with modelling natural phenomena, where data typically consists of changes in a phenomenon. Position-to-term generalisation is a plausible pre-concept towards understanding that relations between two sets of numbers might (sometimes) be expressed as general algebraic 'rules'. However, in spatial sequence tasks the position number may not be understood as a variable but merely as a label, so we do not assume that such tasks provide information about students' understanding of variables.

For the following geometric pattern, there is a chain of regular hexagons (meaning all 6 sides are equal):



1.
For **1** hexagon the perimeter is **6**.
For **3** hexagons the perimeter is **14**.
For **2** hexagons the perimeter is ______________
For **5** hexagons the perimeter is ______________
Note: perimeter is the number of outside edges.

2. Describe the process for determining the perimeter for 100 hexagons, without knowing the perimeter for 99 hexagons.

3. Write a formula to describe the perimeter for any number of hexagons in the chain (it does not need to be simplified).

You can use: $p(n) =$

4. Explain why you think your formula in question 3 is correct.

Figure 1. The task

The task we used (see Figure 1) is a 'typical' spatial sequence task. The task was constructed to allow the students to achieve full generalisation gradually through different kinds of generalisation (Stacey, 1989). The language used was agreed with the teachers to ensure access and familiarity (for example you do not need to remember what perimeter means). With the above considerations in mind we tried to disturb the normal outcomes of such tasks by not providing an ordered data table, thus preventing a quick response of spotting number patterns. We did not expect, therefore, to replicate the findings in which students try to construct from recursive reasoning. We assumed some familiarity with this task type which is ubiquitous in English schools and which might lead some to assume they need to find a position-to-term relation. It is important to note that the previous survey task involved contextual sequential data and questions about rates of change, so students might be starting the task with sequential strategies in mind.

Population

The survey was given to year 7 to 13 classes from two schools with each school providing data from alternate years. We wanted data from a suitable spread of students in terms of their past attainment and asked each school to use their highest achieving class (called A) and a middle achieving class (called B) in each of years 7 to 11 plus their advanced mathematics classes. The teachers provided random anonymised samples of 10 scripts from each class. In this way we received 20 scripts from each UK year 7 to 11 inclusive, and we also had 10 scripts from years 12 and 13 (total of 120 scripts).

Data analysis

We looked for evidence of all attempts to make relationships between data items, since in these tasks such relations could contribute to understanding functions. The method of analysis was to code each student's responses according to pre-concepts related to functions, whether used correctly or not. We then classified them according to approaches to functional reasoning evidenced across the whole sample. We also coded generalisation types. The analysis process was iterative and comparative.

Student awareness of functionality through generalisation

Analysis led to five categories of functional reasoning:

a.	No answer, often accompanied by "I don't know".
b.	No conceptualisation of functional relationship: Empirical methods involving counting.
c.	A *correspondence approach* to develop a general rule for the relation between the number of hexagons and the perimeter.
d.	A *covariation approach* to coordinate the two varying quantities – number of hexagons and the perimeter – while attending to the ways in which they change in relation to each other.
e.	A *correspondence approach* followed by a *covariation approach*: Expressing a *correspondence approach* when addressing the question of finding the perimeter of 100 hexagons, and then, when asked to generate the formula for any number of hexagons, moving to *covariation approach*.

Examples:

Category		Response to Q2	Response to Q3&4	Year/ group
1	b	You can count up in 6 until you get to 100	No response	7 B
2	b	You count how many edges in the chain	No response	9 A
3	c	Because the edges join up they all link together but the 2 end hexagons have 5 sides so you do 5*2 then the hexagons in the middle have 4 sides so you do 98*4 then add the two together	2*5, then how many hexagons in the middle of the end ones * 4. Because if you had 3 hexagons together the end ones have five sides 5*2 then the middle one has 4 so 1*4 then add the answers together = 14.	8 B
4	c	You would have to multiply 6 (the number of sides) by 100 (number of hexagons) = 600, because some sides are joined you have to take them away	6*(number of hexagons) = [space] – number of joint sides. Because if you multiply the number of sides by number of hexagons and then subtract the number of joint sides it will be correct.	10 A
5	c	Each edge hexagon has 5 outside edges, and each hexagon between has 4. Therefore out of 100, 2 would contain 5 and 98 would have 4. If p = perimeter and x = number of hexagon, p=4x+2.	p = 4x+2. I can check it using values which I already know, like 1 and 2. P = 4 * (1) + 2 = 6 for 1. P = 4 * (2) +2 = 10 for 2.	13
6	d	For the amount of hexagons, if you go up by one the perimeter goes up by four.	4n+2. If you do 1 * 4 + 2 that = 6 and that is your answer	8 A
7	d	You are adding 4 sides every time you add a hexagon, so you would multiply 4 by 100 then minus 4 and add 6 for the first hexagon	((4n) - 4) + 6. Because it works for all the numbers used in 2.1	10 A
8	d	4n + 2. You can see the pattern of the perimeter value. They increase by 4 every time for every 1 hexagon! but start at 6 (for 1 hexagon)	4n+2. The jump is 4 every time but it doesn't start at 0 it starts on 6 which is 2 more than 4	13
9	e	1 hexagon is six so if you times 6 by 100 it is 600 that is how much the perimeter would be	Every 1 hexagon you add the perimeter goes up 4. Because one hexagon is 6 and two hexagons is 10 but then 3 hexagons is 14 these numbers are going up by 4	8 B
10	e	Get one hexagon and times it by a hundred	p(n) = n*4.Because the perimeter increases by 4 for	9 B

			every 1 hexagon so it is number of hexagons multiplied by 4	
11	e	To find the perimeter of 100 hexagons you can multiply the perimeter for 1 hexagon by 100. This works as it's in proportion. Example: 1 hexagon – perimeter of 6. 100 hexagons = perimeter of 600	p(n)=(n)-1.For every shape added, one side of the hexagon is lost	11 A

Type of generalisation

Each student's response was categorised according to its generalisations: (1) no correct generalisation of any kind; (2) generalisation expressed correctly in verbal terms only, or (3) generalisation expressed correctly verbally as well as algebraically. Thus the response in example #1 was coded as (1); the response in example #3 was coded (2); example #5 was coded as (3).

Results

Table 1 presents the distribution of the approaches within the A classes. We quantified outcomes across years and groups in order to see if there is any evidence of progression towards successful generalisation, or variation in approaches, bearing in mind that our sample is too small to make generalisations and what we are looking for are conjectures about development. As shown in Table 1, the *correspondence approach* to conceptualising the functional relationships was the most common within the A classes (57%) with many younger students making the proportional assumption referred to by Stacey in an effort to state a position-to-term rule (1989). The *covariation approach* was less widespread (27%). The *correspondence approach* followed by a *covariation approach* constituted 14% of the responses. The categories are distributed among years in a rather 'messy' form, with no specific pattern or order, with the exception of the first two categories of absence (1) and pre-functional approach (2) which are marginal and are expressed in early years only. The *correspondence approach* was the most common with the B classes as well (46%) with the *covariation approach* and *correspondence + covariation approaches* both at 14% (B classes not shown in Table).

Table 1: distribution of the approaches within the A classes

A classes	UK07A	UK08A	UK09A	UK10A	UK11A	UK12	UK13	Total
No answer	0	0	1 (1,0,0)	0	0	0	0	**1 (1%) (1,0,0)**
No conceptualization	0	0	1 (1,0,0)	0	0	0	0	**1(1%) (1,0,0)**
Correspondence approach	6 (6,0,0)	3 (2,0,1)	5 (3,0,2)	4 (2,0,2)	7 (2,2,3)	8 (1,1,6)	6 (0,0,6)	**39 (57%) (16,3,20)**
Covariation approach	1 (1,0,0)	5 (3,0,2)	1 (0,0,1)	4 (0,0,4)	2 (0,0,2)	2 (0,0,2)	4 (0,0,4)	**19 (27%) (4,0,15)**

Correspondence then covariation	3 (3,0,0)	2 (2,0,0)	2 (2,0,0)	2 (2,0,0)	1 (1,0,0)	0	0	**10 (14%) (10,0,0)**
Total	**10 (10,0,0)**	**10 (7,0,3)**	**10 (7,0,3)**	**10 (4,0,6)**	**10 (3,2,5)**	**10 (1,1,8)**	**10 (0,0,10)**	**70 (100%) (32,3,35)**

The triples in the cells of Table 1 show the distribution between generalisation types (1), (2) and (3). As with another task (Ayalon, Lerman and Watson, 2013) results from the A groups have indications of progression towards full generalisation (see bottom row). B group results are dominated by no correct generalisation across years, suggesting perhaps grouping, school or teaching effects that require further probing.

We relate approaches to success in achieving a correct generalisation, since expressing relations symbolically is also a precursor to functional understanding. On the basis of this small data set we can conjecture about the strongest connection between method and success being with the co-variation approach, and this does not accord with most other studies. 15 out of the 19 who tried it in A classes were successful (in contrast to 20 out of 39 within the correspondence approach). In B classes only 2 students reached successful generalisation at all, and they used correspondence. Of both groups, those who did not succeed with co-variation failed because they did not take starting values into account, as Dörfler (2008) and Radford (2008) point out. Those who did not succeed with the correspondence approach either assumed proportionality or took a deconstructive approach but did not deal adequately with the subtraction required (see example 4).

Discussion

Our data appears to contradict to some extent other studies in two main ways. The non-sequential presentation of data appears to have prevented some over-dependence on a recursive approach that focuses solely on differences in output terms. Instead, it is plausible that when a sequential approach requires some deliberate reorganisation of data, it is more likely to be associated with successful identification of co-variation and its role in constructing full generalisation. A correspondence approach, which is reported as not being usually the first resort, was indeed the first resort of about two-thirds of our students. This suggests that correspondence is a concept available for students so long as they are not immediately distracted by sequential features, and also that past experience has an effect although we presented the data in an unusual form. Our approach to analysis therefore offers three features of students' pre-functional approach to data: willingness to look for correspondence; willingness to explore co-variation; and flexibility in combining those approaches. Our initial analysis of the previous task which was numerical, contextual and with sequential variations (not yet published) showed similar propensities. When correct symbolisation is added to the mix, we note that co-variation had a higher success rate than correspondence among the relatively high attaining students.

Although this task and its analysis are only a small part of our whole project, and progression and success is seen clearly only in the A groups, we conjecture that further research about students' search for co-variation and their understanding of non-sequential data might reveal more pre-functional strengths than are shown in these typical generalisation tasks. The decisions we made about presentation have, we believe, enabled students to show co-variational understanding. We also need to find out more about how generalisation in these tasks was emphasised in the schools in order to understand the difficulties experienced by the B classes, and there is a need to consider the role of sequential generalisation tasks in the curriculum as our data suggests they can be used as a pre-conceptual pathway towards gradient in contrast to the usual curriculum function of expressing generality. The next stage of our research will be to compare the UK findings with the Israeli findings.

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