An analysis of the essential difficulties with mathematical induction: in the case of prospective teachers

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Although proof by mathematical induction (MI) is one of the important methods of mathematical proof, gaps and difficulties have been reported in mathematics education research so far. This study provides an analysis of the essential difficulties with mathematical induction that are experienced by prospective mathematics teachers. We take the notion of "mathematical theorem" proposed by an Italian research group, and use this to describe in more detail the structural understanding of MI from a theoretical standpoint. Data are collected by a set of questions based on the idea of "proof script" method. The results suggest that the difficulties of MI are concerned with prospective teachers' understanding of logical relations which we call "sub-theorem" or "meta-theorem".

Keywords: mathematical induction; mathematical theorem; prospective teacher

Introduction

In many countries, MI has been introduced at upper secondary school level, although in some countries it may be intended to teach at college or university level. A number of the previous studies on MI in the field of mathematics education have investigated various difficulties or weak understanding targeting high school students, university students, or prospective teachers (e.g., Ernest, 1984; Dubinsky, 1986, 1990; Dubinsky & Lewin, 1986; Movshovitz-Hadar, 1993; Stylianides, Stylianides, & Philippou, 2007; Palla, Potari, & Spyrou, 2012). In most of them, it can be seen that the main claims have been shown as either some empirical investigations, or the development of teaching materials and/or method. However, only a few studies have paid attention to a theoretical framing for identifying causes of these difficulties beyond the literature review (although there are possibly some exceptions like Dubinsky and Lewin (1986) or Harel (2002)). Since different difficulties have been reported in different studies, a broader theoretical framework may be necessary to characterize different kinds or levels of difficulties with MI more explicitly. Recently, Palla et al. (2012) mentions that there is a gap between the operational and structural level in understanding MI, as follows:

> Operational level is the initial approach to MI, which is also emphasized in most school textbooks. At this level, the structure of the natural numbers is implicit and appears in an intuitive form. The "structural level" is mainly encountered in advanced mathematical studies and refers explicitly to Peano's fifth axiom of the structure of natural number. (Palla et al., 2012, p. 1025)

We do recognize the essential importance of the structural understanding of MI. However, it seems that the concept of "structural understanding" is rather ambiguous due to different structural dimensions of the proof by MI, and thence there is a room for considering a theoretical framework which allow us to conceptualize the different structural dimensions of MI more explicitly.

This paper aims to propose a theoretical model of proof by MI, based on the idea of "mathematical theorem" (Mariotti, Bartolini, Boero, Ferri, & Garuti, 1997), and conceptualize the structural understanding of mathematical induction in terms of the proposed model through the investigation of prospective mathematics teachers' difficulties of MI.

A theoretical modelling of proof by mathematical induction

In general, a proposition "$\forall n \in \mathbf{N}$, P(n)" can be proven by the two steps in MI: the base step, which establishes the base case such as P(1), and inductive step, which proves the implication P(k)→P(k+1) for an arbitrary $k \in \mathbf{N}$. Since both the base and inductive step have been performed, by appealing to the Principle of Mathematical Induction (Peano's fifth axiom for the foundation of natural numbers), the original proposition P(n) holds for all natural numbers. From the logical point of view, by appeal to logical inferences such as *conjunctive inference* (p, q →p∧q) and *modus ponens* ([p, p→q]→q), the structure of proof by MI can be represented as follows:

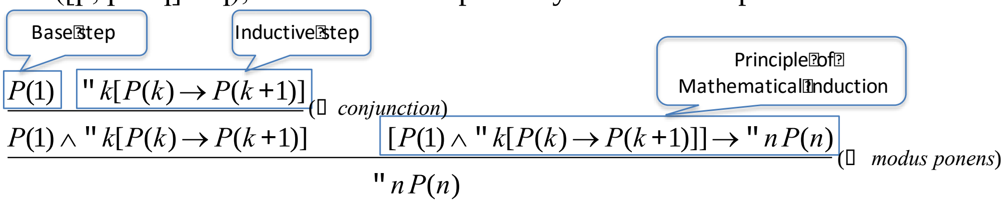

Figure 1: Logical inference form of MI

In order to elucidate difficulties of students' understanding of MI, we shall conceptualize this logical structure in terms of the notion *mathematical theorem* (Mariotti et al., 1997; Antonini & Mariotti, 2008), where interrelationship between *statement*, *proof*, and *theory* can be taken into account. According to the characterization by Mariotti *et al*. (1997), a *mathematical theorem* consists of a system of relations between a *statement*, its *proof*, and the *theory* within which the proof makes sense. Indeed, in mathematicians' mathematical practice, a mathematical assertion such as a proposition and its validation is always considered in a certain theoretical context such as geometrical, arithmetic, algebraic, and other contexts. According to Antonini and Mariotti (2008), when referring to the triplet constituted by *statement*, *proof*, and *theory* as (S, P, T), "there are no limitation on the type of proof (direct, indirect, by induction, etc.)" (p. 404). Moreover, they elaborated this triplet for the sake of analyzing the logical structure of indirect proof, and they introduced the notions of *meta-theorem* constituted by *meta-statement*, *meta-proof*, and *meta-theory*. Although they have not mentioned MI, the notion mathematical theorem can be applied to the logical structure of MI. Let us consider an example in which a proof by MI is provided.

S: The following equation holds for $n \in \mathbf{N}$: $1+3+5+7+\cdots+(2n-1)=n^2$ (☆)

P: *Basis:* show (☆) holds for n=1.

In the left side of equation, the only term is 1, and in the right side of equation, $1^2=1$.

Thus it has been shown that (☆) holds for n=1.

Inductive step: show that if (☆) holds for $n=k$, then it also holds $n=k+1$.

Assume (☆) holds for $n=k$, that is, $1+3+5+7+\cdots+(2k-1)=k^2$. Add $2k+1$ in the both sides of equation. Itmust then be shown as follows.

$1+3+5+7+\cdots+(2k-1)+(2k+1)=k^2+(2k+1)=(k+1)^2$

The above equation shows that (☆) holds for $n=k+1$.

Since both the basis and inductive step have been performed, (☆) holds for all natural numbers.

In the example above, in order to prove an original statement S, proofs of two different statements are given: "(☆) holds for n=1" and "if (☆) holds for n=k, then it also holds n=k+1". We will call these statements given in the basis and inductive step *sub-statements*, and label these two *sub-statements S** and *S*** respectively. We also call the two proofs of sub-statements *sub-proofs P** and *P***. In the sub-proofs, equalities or polynomial identities (multiplication formula), are applied as *a theory T*. Thus, we can interpret both the basis and inductive step as two kinds of triplet of sub-theorem (S*, P*, T) and (S**, P**, T). The final sentence, that is, "Since both the basis and inductive step have been performed, (☆) holds for all natural numbers" is a result of application of the Principle of Mathematical Induction (PMI), that is also considered as *a theory T*. However, the appeal to PMI is usually implicit in the proof by MI. We therefore consider the implicit nature of PMI as *meta-theoretical* status, which is concerned with the logical structure of MI mentioned above.

From the logical point of view, the proof of S* and S** can be represented as (S* ∧ S**) by simple logical inference (i.e., *conjunctive inference*), and PMI can be represented as ([S* ∧ S**]→S). In this case, it is possible to derive the validity of the original statement S from (S* ∧ S**) and ([S* ∧ S**]→S) by appealing to *modus ponens*. Referring to Antonini & Mariotti (2008), we call the statements (S* ∧ S**) or ([S* ∧ S**]→S) *meta-statement*, the proofs of ([S*, S**]→S* ∧ S**) or ([(S* ∧ S**), ([S* ∧ S**]→S)]→S) *meta-proofs*, and the inference rule (the logical theory), in which the *meta-proof* makes sense, *meta-theories*. Thus, we now arrive at the triplet of *meta-theorem* constituted by *meta-statement*, *meta-proof*, and *meta-theory* as (MS, MP, MT). As a result, Figure 1 showed above can be reconstructed as a structure of *meta-theorem* specific to MI as Figure 2.

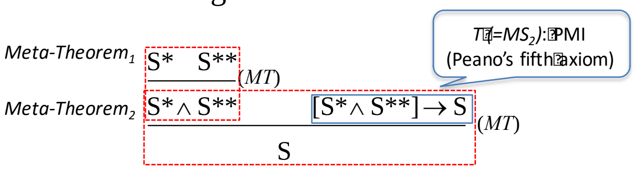

Figure 2: Structure of meta-theorem of MI

The first meta-theorem (MS_1, MP_1, MT_1) is constituted by the meta-statement MS_1(S ∧ S**), the meta-proof of MS_1, and the meta-theory (*conjunctive inference*). The second meta-theorem (MS_2, MP_2, MT_2) is constituted by the meta-statement ([S* ∧ S**]→S), the meta-proof of MS_2, and the meta-theory (*modus ponens*).

Based on the analysis above, in proving S by MI, we can recognize two sub-theorems as [(S*, P*, T), (S**, P**, T)], and two meta-theorems [(MS_1, MP_1, MT_1), (MS_2, MP_2, MT_2)]. When writing or reading the proof by MI, one may often pay attention to the sub-theorems but rarely recognize the substance of the meta-theorems because of its implicit nature. Therefore, from the comprehensive point of view, there is an essential gap between the sub-theorems and the meta-theorems.

Method

Data are collected by a set of questions based on Stylianides et al.'s (2007) item with additional input from the idea of "proof script" (Zazkis & Zazkis, 2015), which involves a scripted proof and a scripted dialogue. We use this method as a tool for engaging prospective teachers in considering particular students' difficulties as well as for identifying the prospective teachers' comprehension of the proof.

The figure 3 shows a scripted proof (and a given statement) used in the present study. It is fairly based on the item developed by Stylianides et al. (2007). In this script, the proposed proof is invalid, but there are three points that have to be considered. Firstly, the given statement, an equation, does not hold for any natural numbers. Secondly, the base step is missing in the given proof. Thirdly, the inductive step is still correctly applied. We, like Sylianides et al. (2007), aimed to see whether the prospective teachers who would realize the absence of the base step would be able to explain the necessity of the base step. We also intended to investigate the prospective teachers' understanding of the logical validity of the inductive step.

Statement: For every $n \in \mathbf{N}$ the following is true: $1+3+5+\ldots+(2n-1)=n^2$ (*)

Proof: I assume that (*) is true for $n=k$: $1+3+5+\ldots+(2k-1)=k^2+3$

I check whether (*) is true for $n=k+1$:

$1+3+5+\ldots+(2k-1)+(2k+1)=(k^2+3)+(2k+1)=(k^2+2k+1)+3=(k+1)^2+3$

True.

Therefore (*) is true for every $n \in \mathbf{N}$.

Figure 3: A scripted proof (Stylianides et al., 2007, p. 151)

In order to utilize this item, unlike Stylianides et al. (2007), we introduced the questions with the following dialogue (Figure 4). The first dialogue by Alan and Barbara is concerned with the reason way the base step is essential. The second dialogue by Christine and David is related to the logical validity of the inductive step, although David's remark might suggest additional misunderstanding about circular reasoning (Ernest, 1984). Participants were asked to first consider the scripted proof above and then write their thought or rationale of four each scripted dialogue.

Alan and **Barbara**, high school students, are having a conversation about the above proof. Read through and answer the following questions.

Alan says: *This proof is not valid. Because its first step is missing*.

Barbara says, followed by Alan: *Why is it necessary to check for n=1?*

Christine and **David**, high school students, are having a conversation about the above proof. Read through and answer the following questions.

Christine says: *This proof shows the inductive step, that is, "if it is true for n=k, then it is true for n=k+1". So, the proof of inductive step is valid.*

David says, followed by Christine: *Mathematical induction is the method in which you assume what you have to prove, and then prove it. So, I have a suspicious likeness to assuming what you have to prove!*

Figure 4: A scripted dialogue

The 38 prospective secondary school mathematics teachers both in England (N=19) and Japan (N=19) were asked to write their thoughts by reading the above scripted proofs and dialogues. The 19 participants from England were trainees on a Post Graduate Certificate of Education in secondary mathematics course. Most of them have majored in mathematics at the undergraduate level, although a few majored in physics or engineering. The 19 participants in Japan were third year undergraduate mathematics course students in the faculty of education. In this paper, as space is limited, we have concentrated on participants' responses to Alan and Barbara's dialogue, and paid scant attention to the responses to Christine and David's dialogue.

Results and discussion

Results

Most of participants (89.5%; 34/38) agreed with Alan's remark stating, for example, "this proof is not valid. Because its first step is missing", although three participants disagreed and one participant's answer was unclear. However, it does not mean that most of participants were able to explain the reason why the base step is necessary. For example, J5 (participant #5 in Japan) wrote an acceptable explanation, but he/she also remarked that the base step was unnecessary as follows: (underline is added):

> J5: When it says "for n=k", it doesn't say that k is an arbitrary natural number. But if k is a natural number, I don't think that it needs to prove the case for n=1.

Some of them were limited to saying that the statement (a given equation) is not true for n=1. For example, E19 and J6 wrote as follows:

> E19: There is no evidence that $1+3+5+(2k-1)+(2k+1)=(k^2+3)+2k+1$. How did they get this?

> J6: When prove for n=1, we see that the statement is not true.

Indeed, the given statement does not hold for any natural numbers, although in the presented proof the inductive step is still correctly applied. In this sense, these written responses suggest the difficulty exists in understanding of the inductive step as well as recognising the base step is an essential part of the proof. Especially, it indicates weak understanding of the inductive step as an implication statement, or weak understanding of the meaning of the inductive hypothesis. With regard to this, different difficulties were provoked when explaining the necessity of the base step. The participant J1 seems to believe that the inductive step can prove that both P(k) and P(k+1) are true. Similarly, E12 seems to think that the presented proof showed P(k+1) rather than $P(k) \rightarrow P(k+1)$.

> E12: if the case is true for n=k+1 and for n=1 then it must be true for all cases of n=1, 2, 3, ….

> J1: If it is true for n=k and n=k+1, it contradicts the fact that it does not hold for n=1. So, this proof is not true.

These difficulties are related to the characteristics of the implication as an object (Dubinsky, 1986, 1990; Stylianides et al., 2007), through the assumption of its antecedent (i.e., viewing "$p \rightarrow q$" as an object). In fact, the participant J1 also explicitly exposed the misunderstanding when responding the Christine's remark as follows:

> J1: We have to assume that antecedent is true, if not, the statement will always be false.

Discussion

Results show that most of the prospective teachers participating understood the meaning of scripted statement and proof, and they realized that the base step is missing and recognized the validity of the inductive step. However, results also show that there are three main difficulties in explaining why the base step is necessary and the logical validity of the inductive step.

1) Weak understanding of the necessity of the base step
2) Difficulty with distinguishing the truth of the original statement with the truth of the implication statement (the proof of the inductive step)
3) Difficulty with recognizing the implication statement as an encapsulated object and/or weak understanding of the inductive hypothesis

Here we pay special attention to the third difficulty, because this difficulty is concerned with the distinction between the sub-theorem and the meta-theorem. Most of the participants accepted the presented proof but they faced difficulty in explaining the truth of the implication statement. It can be considered for such participants that their understanding of MI is based on sub-theorem. The following findings suggest the essential difficulties with becoming aware of meta-theorem.

- Applying an incorrect implication rule such that: If P(1) and P(k), and if P(k) implies P(k+1), then for P(n);
- Thinking the truth of the antecedent in advance, in order to prove the implication statement;
- Believing that the inductive step establishes the truth of P(k+1) rather than P(k)→(P+1).

One of the contributions of this study was that we made the distinction between the difficulties that occurred in the sub-theorems and the difficulties that occurred in the meta-theorems, and discussed the essential gap between them.

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