A brief history of quadratic equations for mathematics educators

Leo Rogers and Sue Pope

British Society for the History of Mathematics (BSHM); Manchester Metropolitan University

In contrast to the 2007 secondary curriculum, the new English mathematics curriculum alludes only to Roman Numerals in the primary programme of study. Despite the words: 'Mathematics is a creative and highly inter-connected discipline that has been developed over centuries, providing the solution to some of history's most intriguing problems.' in the Purpose of Study section there is no further mention of historical or cultural roots of mathematics in the aims, or in the programmes of study. The increased expectations for lower and middle attainers in the new curriculum, challenge teachers to make more mathematics accessible and memorable to more learners. The history of mathematics can provide an engaging way to do this. There are also opportunities in post-16 mathematics. We use quadratic equations to illustrate some of the ways that history of mathematics can enrich teaching of this topic.

Keywords: history of mathematics; quadratic equations; BSHM

Introduction

In the 2014 National Curriculum for mathematics (DfE, 2014) all KS4 students are expected to solve quadratic equations and interpret graphs of quadratic functions. The history of solving quadratic equations offers an accessible way of introducing this content that is more likely to capture students' interest.

The benefits of using the history of mathematics in the classroom are well documented (Fauvel and van Maanen, 2000):

- Enhances mathematical learning
- Exposes the nature of mathematics and mathematical activity
- Enriches teacher's understanding and pedagogic approach
- Promotes an affective predisposition towards mathematics
- Reveals mathematics as a cultural-human endeavour

and it can provide:

- Starting points for children to ask their own questions (inquiry based learning, philosophy for children)
- A means for making links with other aspects of the curriculum: citizenship, geography, history, science, art and design, design and technology
- Opportunities for connecting different mathematical ideas

There are several articles on the Mathematics Millenium project website aimed specifically at teachers (Rogers, n.d.a, Rogers, n.d.b and Robson, n.d.)

In this short article, we introduce some key resources and explore the development of associated mathematical ideas.

A map of the development of quadratic equations

The history of the solution of quadratic equations extends across the world for more than 4000 years. The earliest known records are Babylonian clay tablets from about

1600 BCE where the diagonal of a unit square is given to five decimal places of accuracy. The map below illustrates the various sources. Mathematical and technical knowledge was transmitted, applied and modified by travellers across the continents (Katz, 2007; Robson and Stedall, 2009) and (Fauvel and van Maanen, 2000).

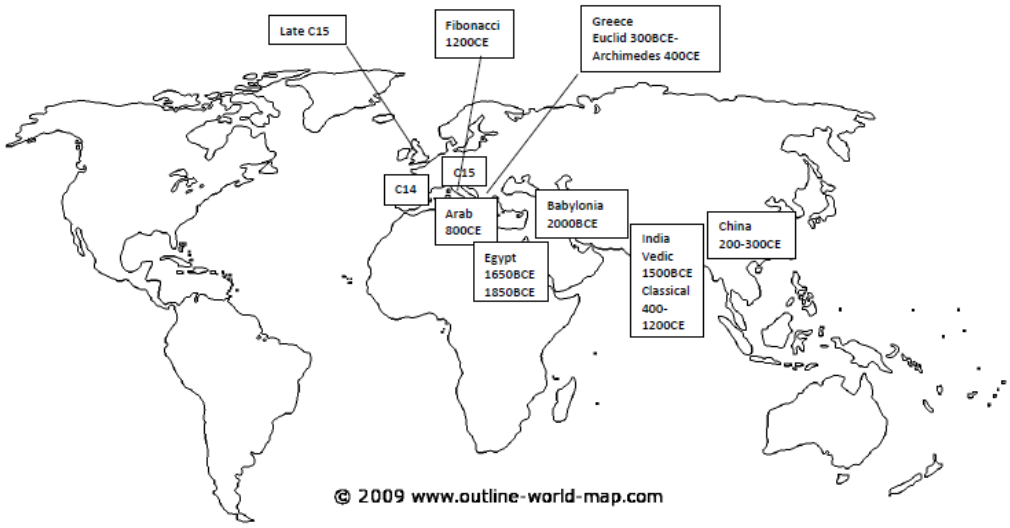
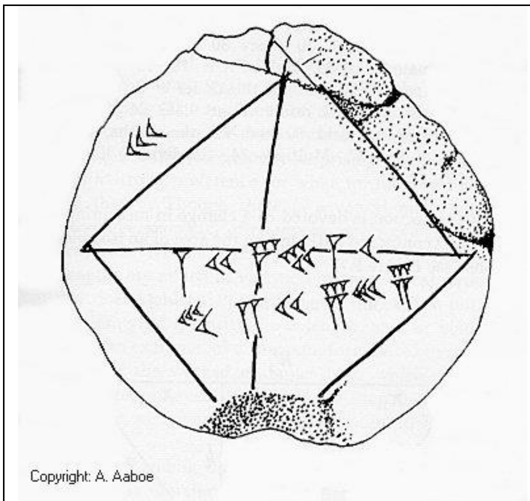


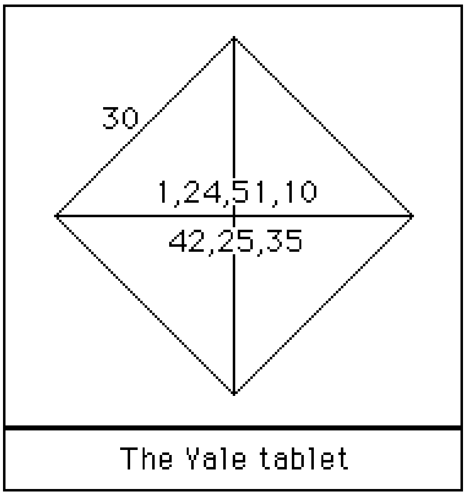
Figure 1. Map showing the historical and cultural roots of quadratic problems

The approach to quadratic equations taken today is relatively modern. Indeed, the use of algebraic symbols only began in the 15th century.

Solving quadratic equations

The Babylonian clay tablet below is a valuable and accessible source suitable for working on different number bases (Babylonians used bases 10 and 60), as well as on decimal approximations to irrational numbers and quadratic problems (Burns, 1997).





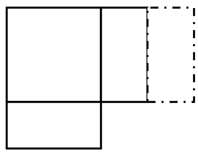
$1+\frac{24}{60}+\frac{51}{60^2}+\frac{10}{60^3}=1.41421 \text{ to } 5 d.p. \approx \sqrt{2}$

$42+\frac{25}{60}+\frac{35}{60^2}=42.4264 \text{ to } 4 d.p. \approx 30\sqrt{2}$

Figure 2. The Yale tablet (1600 BCE) with transcription into modern notation and translations

This prompts the question – 'how did they do it?' The original mind-set was entirely geometrical, where instructions were given in the language of the culture, and the

operations described a series of ‘cut and paste’ actions. These sets of instructions were clearly ‘algorithmic’ in their nature. Today, this approach can lead to useful explorations with a spreadsheet for an approximation to the square root of two.
Any rectangle of area two can be adjusted to look more like a square by splitting the surplus of the square into two and moving one to the other edge, so a one by two rectangle becomes a square of side 1.5 less a square of side 0.5. This suggests another rectangle of area two with dimensions 1.5 and four thirds. The process is then repeated. The table illustrates successive dimensions.



length	width	correction
2	1	0.5
1.5	1.333333333	0.083333333
1.416666667	1.411764706	0.00245098
1.414215686	1.414211438	2.1239E-06
1.414213562	1.414213562	1.59484E-12
1.414213562	1.414213562	-1.11022E-16

Figure 3. A possible iterative approach to estimating the square root of two

The approach converges rapidly and can be generalised to finding an approximation to any square root. What mathematics is needed in this approach?

Determining the dimensions of a square with the area of a rectangle was important in the development of early geometry and the Greek mathematicians in 300BCE were familiar with this. Figure 4 is from Euclid’s Elements (Book 2, Proposition 14 in Joyce, 1998):

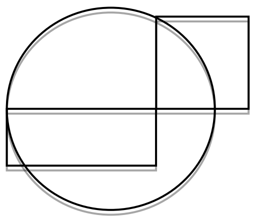


Figure 4. The Greek approach to squaring the rectangle

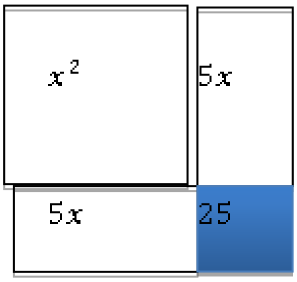
Any polygon can be reduced to a rectangle, using the fact that triangles with the same base and perpendicular height have the same area (Euclid: Book 1, Proposition 36 in Joyce, 1998), so any polygon can be reduced to a square. The classic problem of ‘squaring the circle’ originates from Greece. In the 3rd century CE *Archimedes* in Alexandria and *Liu Hui* in China demonstrated that the area of a circle is proportional to the square of its radius.

In the 8th century the Caliph Al-Mansour sent people ‘as far as China’ to collect knowledge from the known world for his ‘House of Wisdom’ in Baghdad. From this ‘research centre’ by the end of the century Al-Khwarizmi described solutions for six types of mercantile and inheritance problems where the quantities were all treated as rational numbers, see table 1.

This is often regarded as the beginning of algebra which takes its name from ‘Al-jabr’ which means ‘restoration’, or adding a number to both sides of the equation. Arab mathematicians showed that the ‘algebraic’ instructions were equivalent to the Euclidean procedures for finding areas.

Al- Kharizmi's definition	Modern notation
Squares are equal to roots	$x^2 = bx$
Squares are equal to numbers	$x^2 = c$
Roots are equal to numbers	$x = c$
Roots and square are equal to numbers	$x^2 + bx = c$
Square and numbers are equal to roots	$x^2 + c = bx$
Roots and numbers are equal to squares	$bx + c = x^2$

Table 1. Al-Khwarizmi's Six Types of Problem



The solution is this:
You halve the number of roots, which in the present instance yields five.
This you multiply by itself, the product is twenty-five.
Add this to thirty-nine, the sum is sixty-four.
Now take the root of this, which is eight.
Subtract half the number of roots, which is five, the remainder is three.
This is the root of the square which you sought, the square itself is nine.

Figure 5. Ten roots and the square are equal to 39

At the time, all mathematical problems had to have positive solutions. The methods came into Europe by two routes: one from Mesopotamia through the Mediterranean and Italian merchants, where the methods (described in local languages) are found in *Fibonacci's* collection, the *Liber Abaci* (1202) and the other route through North Africa and Arab Spain by the Latin translators in Toledo in the 12th and 13th centuries (Katz & Parshall. 2014: 177-178).

The Development of Notation

By this time, mathematicians had tackled problems where they developed techniques for dealing with special problems that involved surds $(3 + \sqrt{5})$, roots of surds $\sqrt{3 + \sqrt{5}}$ and some special cubic equations but no general approach was available at this time.

As printing technology became available, the Latin texts were translated; first into Italian, Spanish and German, and then to the rest of Europe. By the late 15th century symbols started to be used for operations and in the 16th century symbols were used to represent arithmetical objects and operations on them.

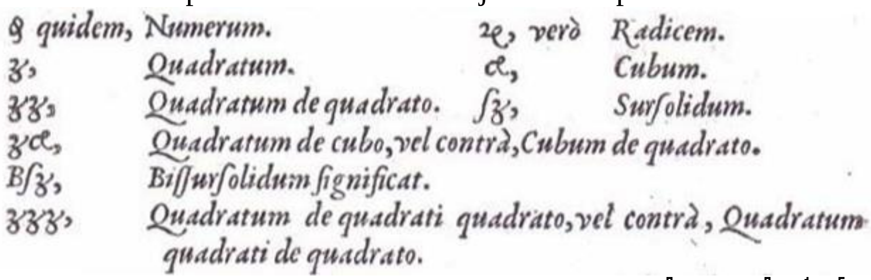


Figure 6. German 'cossic' notation 1551 'Number' x, square x^2, cube $x^3, x^4, x^5, x^6, x^7, x^8$.

About the same time the first equations appeared with symbols for relations and we find the famous equals sign of Robert Recorde, described as "a pair of parallels because no two things can be more equal."

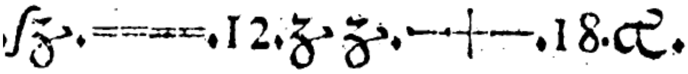


Fig 6. Recorde 1557 Whetstone of Wit: Cossic Numbers, $x^5 = 12x^4 + 18x^3$

In 1545 the Italian, Girolamo Cardano published his *Ars Magna* (The Great Art, or the Rules of Algebra) where he used the work of his predecessors and the same basic procedure that had been handed down from Mesopotamia to solve a whole series of equations. This included where the roots were non-rational (surds) and in the process he discovered strange new numbers for roots, which he called imaginaries. Cardano explored the problem: "What pair of numbers have a sum of ten and product of 40?" We might write the problem as $x(10-x) = 40$, but he had no such notation. Working almost entirely in Latin (the academic language of the time) he had only p for plus and m for minus and a square root symbol R, so described his result as: $5p{:}\,R{:}\,m15$ and $5m{:}\,R{:}\,m15$ which in modern notation is $5+\sqrt{-15}$ and $5-\sqrt{-15}$. Cardano suggested you "suspend belief" and work with the square root of negative 15 to find the two distinct roots. By the end of the century another Italian, Rafael Bombelli gave a comprehensive view of the algebra of the time, giving all the rules for operations on "imaginary numbers". (Katz & Parshall. 2014: 229-235) By the time of Descartes (1596 - 1650), and certainly by the late 17th century algebra was written in virtually modern notation that most 14 year olds can read.

Every quadratic has two roots

The fundamental theorem of algebra was articulated by Albert Girard in 1629 as "Every algebraic equation .... admits of as many solutions as the denomination of the highest quantity indicates ...". No proof was offered. It was the late 18th century when mathematicians, including d'Alembert and Gauss, developed proofs (https://www.encyclopediaofmath.org/index.php/Algebra, fundamental_theorem_of). The link between equations and graphical representations was due to Descartes. This connection is important for students today, where realising that every point on a curve represents an instance of an equation is a fundamental part of an understanding of functions (Watson, Jones and Pratt. 2013: 176-199).

Conclusion

The social and cultural aspects of the origins of mathematics lie in solving practical problems. During the 8th to 10th century CE the Arab civilisation collected mathematical and scientific knowledge from around the known world. They made the connection between what we know now as algebra and the geometric tradition that has its origins in Mesopotamia (Babylon). Al-Khwarizmi, introduced the notion of the unknown and classified the problems to be solved.

Quadratic equations have an exciting history (Budd and Sangwin, 2004a) and are important in modern society too (Budd and Sangwin, 2004b). We hope this short article will inspire colleagues to incorporate the history of mathematics when teaching quadratic equations. For further reading we recommend an excellent, accessible short history by Stedall (2012).

Context

This short article arises from the BSRLM History of Mathematics working group as part of its preparation for an Anglo-Danish History of Mathematics in Education conference to be held at Bath Spa University, 21-24 August 2016. http://www.bshm.ac.uk/events/history-mathematics-education-anglo-danish-collaboration.

References

Budd, C. & Sangwin, C. (2004a). *101 uses of quadratic equations*. Retrieved from https://plus.maths.org/content/101-uses-quadratic-equation.

Budd, C. & Sangwin, C. (2004b). *101 uses of quadratic equations:* part II. Retrieved from: https://plus.maths.org/content/101-uses-quadratic-equation-part-ii

Burns, S. (1997) *The Babylonian Clay Tablet.* Mathematics Teaching 158, 44-45.

DfE (2014). *National Curriculum: Mathematics Programmes of Study.* Retrieved from https://www.gov.uk/government/publications/national-curriculum-in-england-mathematics-programmes-of-study/national-curriculum-in-england-mathematics-programmes-of-study.

Fauvel, J. & van Maanen, J.A. (2000) *History of Mathematics in Education* Dordrecht: Springer Netherlands.

Joyce, D. (1998) *Euclid's Elements*. Retrieved from http://aleph0.clarku.edu/~djoyce/mathhist/mathhist.html

Katz, V. (2007). *The Mathematics of Egypt, Mesopotamia, China, India and Islam.* New Jersey. Princeton University Press.

Katz, V. & Parshall, K., Hunger. (2014). *Taming the Unknown: History of Algebra from Antiquity to the Early Twentieth Century*. Princeton and Oxford: Princeton University Press.

Robson, E. & Stedall, J. (2009). *The Oxford Handbook of the History of Mathematics.* Oxford: Oxford University Press.

Robson, E. (n.d.) Babylonian maths https://motivate.maths.org/content/BabylonianMaths

Rogers, L. (n.d.a) Algebra 1&2; http://nrich.maths.org/6485 and http://nrich.maths.org/6546

Rogers, L. (n.d.b) Trigonometry 1, 2 & 3; http://nrich.maths.org/6843 http://nrich.maths.org/6853&part http://nrich.maths.org/6908

Stedall, J. (2012). *The History of Mathematics: A Very Short Introduction.* Oxford: Oxford University Press.

Watson, A., Jones, K. & Pratt, D. (2013). *Key Ideas in Teaching Mathematics*. Oxford: Oxford University Press.