A look at two algebra tasks involving sequential data, that seem to prompt a scalar rather than function approach to the underlying linear relation

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In this paper we discuss an interview undertaken by one of the authors (DEK) with a group of three Year 8 students and their teacher as part of the design research work of the ICCAMS project. The interview involved two tasks in which pairs of values connected by a linear relationship were presented sequentially, either in a table or as coordinate points on a Cartesian grid. The students were asked to make near and far generalisations, which they tended to do by adopting a scalar (or recursive) approach, either in a step-by-step manner or by chunking. From the interviewer's point of view, the scalar and function perspectives are intimately linked, and on occasions during the interview it was easy to believe that this was true for the students too. However, a closer examination suggests that for these students at least, the connection is still tenuous.

Keywords: algebra; generalising; generic pattern; figurative pattern; sequence

Introduction

School algebra often appears to be about nothing very much. For example, in the 'problem solving' activity (Fig 1) and the Year 7 'mathematical reasoning' example (Fig 2) the letters do not even represent numbers. They are simply used as shorthand for words like 'Months' and 'Rods' - what we have termed *Letter as Object* (Küchemann, 1978).

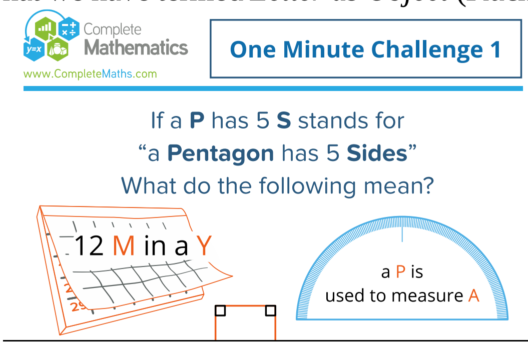

Fig 1. La Salle Education, 2015, p.6

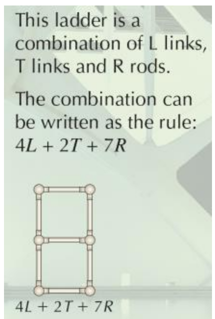

Fig 2. Evans et al., 2014, 1.2 p.42

Where letters do stand for numbers, it can still be difficult for students to make sense of what is going on, or to fathom what the purpose and utility of algebra might be (Ainley and Pratt, 2002). For example, in this extract (Fig 3) from a Y8 spread on 'simplifying', the context is almost entirely spurious: if we are to assume that the first figure is a rectangle (so as to be able to determine the lengths of the unmarked sides), then it would seem reasonable to assume that the 'lozenge' in part **e** is symmetrical, with $t + p = 4t + p$, so that $t = 0$, meaning that two of the sides have length 0. We are given no sense of why lengths have been assigned algebraic expressions or why we might want to know the perimeters - the focus is purely on practising procedures.

Where the algebra content clearly *is* about number, it is often about number as specific unknown (as in Fig 4) rather than generalised number or variable.

Content involving general relations is surprisingly rare. Where students are asked to generalise it is usually in the context of figurative patterns or purely numerical patterns (shown simply as a number sequence, or in the form of ordered pairs of numbers, usually presented sequentially). And usually the underlying relations are linear. In this paper we look at two tasks involving numerical patterns, which we used in an extended interview with three Year 8 students and their teacher.

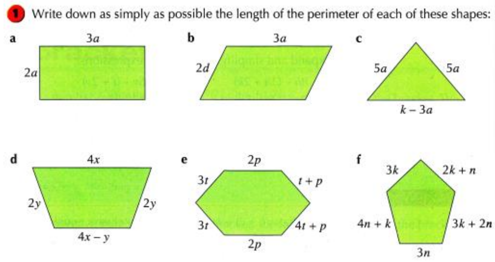

Fig 3. Evans et al., 2002, p.82

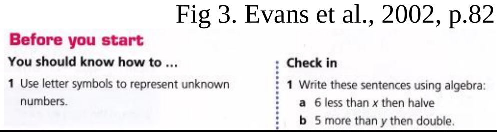

Fig 4. Capewell et al., 2003, p.57

Finding rules

Finding a rule that describes the structure of a figurative pattern or number sequence can be challenging - and engaging. Often, the overall aim of such a task is to express the rule algebraically, by, for example, asking for 'the *n*th term'. Unfortunately, though this is mathematically important, it may not seem purposeful to students, since they may have been successfully finding terms *prior* to constructing the *n*th term.

Fig 5. Hodgen et al., 2012, p.36

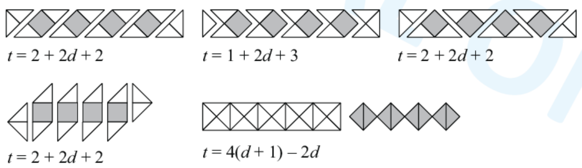

Fig 6.

A useful property of figurative patterns is that it is often possible to describe the structure in different (equivalent) ways. Consider this 'garden path' pattern (Fig 5), which consists of a row of d 'diamond-shaped' tiles surrounded by t triangular tiles. As can be seen from Figure 6, the relationship between d and t can be expressed in lots of ways, which gives such tasks the added purpose of showing that the resulting expressions or relations are equivalent.

Number sequences don't have this richness. Consider this table (right), which involves the same relation as 'garden path' and which essentially represents the sequence 6, 8, 10, 12, ... Assuming the terms continue to increase by 2, then one way to find the *n*th term is to 'add $(n - 1)$ lots of 2 to the first term, 6', giving $t = 6 + 2(n - 1)$. Another might be to 'add n lots of 2 to the 0th term, 4', giving $t = 4 + 2n$. No doubt one could construct other (equivalent) expressions for t, but they are likely to be less easy to come by, and less salient, than those derived from a figurative pattern.

n	t
1	6
2	8
3	10
4	12

In the above table, the relation between n and t is a *function* relation. We can also discern *scalar*, or recursive, relations in the table, in particular the relation 'as n increases by 1, t increases by 2'. As is often the case, this scalar relation is simpler than the function relation, and one aim of our research was to see whether, having spotted a simple scalar relation, this would help or hinder students from seeing the more complex function relation.

Another powerful characteristic of some figurative patterns is that they can be presented 'generically', as in the case of 'garden path' above. There is some evidence (Küchemann, 2010) to suggest that this can help students focus on the function relation between the elements in a pattern (or between an element and the position of the pattern in a sequence). Some of the SMP 11-16 materials (eg, SMP, 1981, p.3) came close to adopting this mode of presentation. However, it is more usual, both in research tasks (eg ZDM, 2008, 40, 1) and in curriculum materials, to see figurative patterns presented as a set of exemplars and one that is ordered (as in the case of 'growth patterns', for example). This makes it far more likely that students will focus on a scalar, or recursive, relation between exemplars rather than on the function relation within the pattern. As suggested earlier, one reason for this is that such a relation is likely to be simpler. However, there is also strong evidence to suggest (eg Vergnaud, 1983; Küchemann, Hodgen & Brown, 2014, p.233) that, for a given level of complexity, students more readily construct relations *within* measure spaces than *between* them.

The interview

The interview, which lasted nearly 50 minutes, took place towards the end of the school year with three students from an 'average ability' Year 8 class. An advantage of such small-group interviews is that the pressure is taken off individual students, who have space to pursue their own lines of thought. At the same time, it is likely that several ideas will be aired at the same time, so that some might be misconstrued or not investigated.

The students, Ja, Jd and Je, would have been about 14 years old. Ja dominated the interview, though Jd was happy to interject, often rather impulsively, while Je's contributions were relatively few but often more considered. We started with the task shown in Figure 7 which was presented to the group on an A5 sheet of paper. Figure 7 also shows what was written on the sheet during the interview, mostly by the interviewer DEK.

In item 1, Ja quickly saw that B would be 27, using the scalar (or recursive) argument that the given values of B in the table go up in 4s. However, he got into a muddle trying to express this, saying

> ... every A is 4... every A is 1... when A is 1, B is 4... when A is 2, B is 8...

At this point the class teacher intervened by reflecting this back:

> As A increases by 1, B increases by 4.

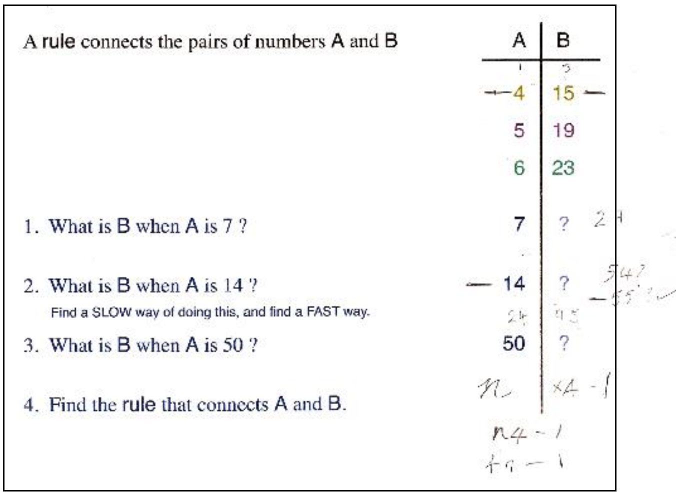

Fig 7.

The teacher's statement might well have been what Ja had in mind initially, and at this early juncture it looked as though Ja might be close to seeing the relation between the values from a function perspective, namely that the value of B is essentially 4 times whatever value A is, with some fixed, minor adjustment (in this case, –1). However, this turned out not to be the case.

Jd arrived slightly late for the interview, while Ja was discussing item 1. Jd immediately said 'I think it's times 4 minus 1'. We decided not to question Jd directly on this rule, so as to see what spontaneous reactions the other two students would have to it. Of course, it would have been interesting to know Jd's rationale - had he just spotted a rule that seems to work, or was it derived from an awareness, as subsequently voiced by the teacher, that 'As A increases by 1, B increases by 4'?

In the event, all three students seemed to treat Jd's rule as a valid rule which they could use to check results gained from their own methods, but not as something that necessarily cohered with these other methods.

Item 2 was designed to see whether the students would fall into the trap of *scaling* the values in item 1, which they duly did ('14 is 2×7, so B is 2×27'). Ja claimed that doubling was just a quick way of 'adding 4':

> My way is adding 4, so... 27... instead of going 8, 9, all the way up to 14, cos I think double 7 is 14, so if I double that [7 to 14], I have to double 27.

Jd agreed. We (DEK) wrote the result, 54, on the worksheet but with a question mark next to it. Ja suggested testing this using Jd's rule, 'Times 14 by 4, minus 1' which resulted in 55, which we also noted down. At this point, Je came up with a curtailed (or 'chunking') version of Ja's scalar +4 method, which she expressed like this:

> To get from 7 to 14... you have to do 7 and then times it by 4, which is 28, and then add the 28 to 27, makes 55.

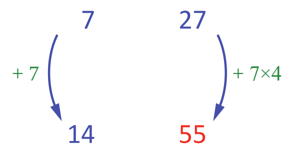

Fig 8.

(Fig 8 shows this schematically.) In the case of Ja and Je, this cast doubt on the scaling method (ie doubling), but Jd still saw doubling as just a faster version of Ja's +4 method.

We didn't manage to resolve the cognitive conflict in the subsequent discussion, so by way of a fresh start, we asked the students whether they could use the given pair of values (4, 15) for (A, B), to find B when A is 14. Ja now adopted Je's curtailed version of his +4 scalar method: in effect, '4 + 10 gives 14, so 15 + 4×10 gives the value of B'. He went on to suggest that we could check this by finding B when A is 24 (another increase of 10). Jd then suggested we could find this value by first finding B when A is 12 and "using Ja's thing of doubling".

The teacher now asked for a far generalisation: "What if A was 215?". Ja replied that we could use the "difference thing" applied to the given pair (5, 19): "Do 210 times 4 and add onto that [19]".

Finally, for this task, we asked, "Say my number here [column A] was *n*?" Jd responded with "'Question mark' equals *n* times 4 minus 1", which matches his original rule, and Ja then wrote '×4 +1'. We pushed this further by asking "Could we write this as an expression using *n*?", to which Je replied "*n*4 take away 1" and then wrote '*n*4 – 1'. Ja rewrote this as '4*n* – 1'.

Our introduction of '*n*' seemed to have immediately prompted a switch from a scalar to a function rule, and quite quickly to a correct standard algebraic expression for the *n*th term. The switch may in part have occurred because it is actually quite difficult to express a scalar rule using *n* [though one could do so in this kind of way: start at A = 5, B = 19, say; then when A = *n*, we can say $n = 5 + (n - 5)$, so $B = 19 + 4(n - 5)$, so $B = 4n - 1$]. The question remains, though, whether the function rule is related to the students' insight into the structure of the numerical pattern (along the lines of 'as A increases by 1, B increases by 4') or whether it is simply a formal expression of Jd's empirically viewed rule, "times 4, minus 1".

To probe this further, we gave students a second, similar number-pattern task, but this time in the context of points in the Cartesian plane. The sheet containing the task is shown below (Fig 9), along with responses written by the students or DEK.

Ja's responses were similar to before, but this time he made systematic use of jumps of 10 in the *x*-coordinate and that this meant a jump of 3×10 = 30 in the *y*-coordinate. For example, to find *m* for the point S(14, *m*), he referred back to the particular given point P(4, 13), and constructed a scalar argument along the lines of '14 = 4+10, so *m* = 13 + 4×30 = 25'. Later, after we had for some reason established that (0, 1) lay on the line through the given points, he used this to find n for the point T(50, n). Thus he constructed another scalar argument along the lines of '50 = 0 + 5 lots of 10, so n = 1 + 5 lots of 30 = 151'. On the other hand, Jd (who hadn't come up with a function rule for this task), was still wedded to scaling. Thus he tried to find n by starting with Q(5, 16) and scaling both coordinates by ×10. Je did the same, thus also getting n = 160, but was more circumspect: "I think I got it wrong".

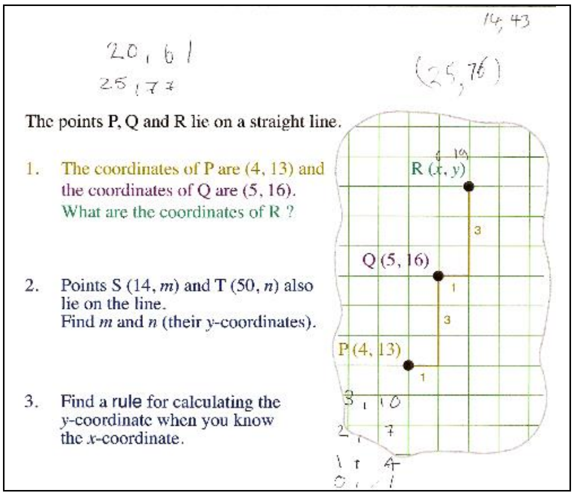

Fig 9.

After Ja had used (0, 1) as a jumping off point to find n in (50, n), we asked whether he could do the same for a point on the line with coordinates (25, ?). He replied that he would prefer to start at (5, 16), as this involved 2 jumps of 10. "But if I *had* to start at (0, 1), I would get to (20, 61) [61 = 1+ 2×30] and add 16 [the y-coordinate of Q(5, 16)]". Of course, this doesn't quite work, and gives 77 rather than 76, though he later managed to correct this.

For this task, the point (0, 1) is the most effective 'jumping off' point for a scalar argument. Expressed formally, the rule looks like this: $y = 1 + (x - 0)\times 3$, which of course is very similar to the standard function form, $y = 3x + 1$.

The question we asked towards the end of the interview seemed to allow Ja (and perhaps the other students) to make this link, though that hadn't been our prime intention. Rather, we wanted to see whether they could use a general scalar argument along the lines of Ja's but without the restriction of only jumping in 10s. So we asked them to imagine someone going from (0, 1) to (25, ?) 'very slowly', one step at a time. "How many 3s would they have added?" Jd started talking about division (he might have been trying to work backwards from the result, y = 76) but Ja intervened:

> Would it be 25 threes? .... When you get to 25 you've gone up 25 3s ... you need to remember to add 1 [points to (0, 1)].

At this point the teacher said, "You've got kind of a rule there in a way. What's your rule?" Ja wasn't quite sure. At first he said, "Is it add 3...? Is it times 3?". A bit later he said "The rule would be add 3 add 1" and then "I was gonna say times 3 add 1 but it wouldn't work". Meanwhile, Je expressed uncertainty about starting at (0, 1), perhaps because it would not allow her to use scaling to get from 0 to 25. "I don't understand how you get anywhere if you start from zero."

We discussed the process again of going from (0, 1) to (25, ?) and DEK drew pairs of lines on squared paper (1 across, 3 up) to show some of the individual steps: "We're going 1 across 3 up, 1 across 3 up, and we're doing that until we've got to 25 across... a long, long, long, long journey." Ja now expressed the rule very clearly, albeit initially using a specific value, 50, for the x-coordinate:

> Say it was 50, you would add on 50 lots of 3. So 50 times 3 is 150, but because you need to remember that 1 you started with [points to (0, 1)], you forgot that 1 at the start, you add 1 to your 150, that's 151.

Finally, Ja said "Whatever x is, you times that by 3 and add 1", which was echoed by Je: "Three x plus add one". The teacher then asked, "We're finding what value, what number?", to which Je replied "y" and then "y equals three x plus one", which was greeted by laughter of recognition by all of us!

Conclusion

The interview supports our view that number-pattern tasks of this sort encourage students to take a scalar rather than a function view of the relations. It also suggests that we can easily underestimate the difficulties students have in seeing the connection between the two viewpoints, when the connection is obvious to us. In turn this suggest that, initially at least, we should make more use of figurative tasks where the pattern can be presented generically - as in this task (Fig 10) from the ICCAMS lesson materials. And when we do use number-pattern tasks, with numbers in a table, the pairs should not always be presented sequentially.

A single row of white tiles is surrounded
by a double layer of grey tiles.
In this pattern there are 10 white tiles.

Imagine the pattern with 50 white tiles.
How many grey tiles will there be?
Use this sketch to help you.

Find different ways of calculating the answer.

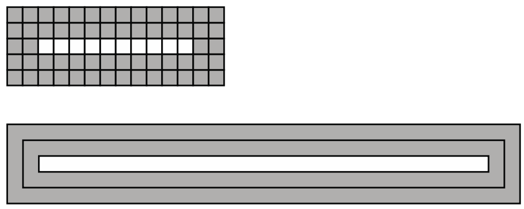

Fig 10. Hodgen et al., 2012, p.37

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