Creating the conditions for children to persevere in mathematical reasoning

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This paper reports on the findings from a small-scale intervention study that explored developing perseverance in mathematical reasoning in children aged 10-11. The interventions provided children with representations that could be used in a provisional way and included opportunities and time to generalise and to form convincing arguments. This enabled the study group to persevere in their mathematical reasoning, from making trials and testing conjectures to forming generalisations and convincing arguments. The children reported pride in their understanding. A tentative framework describing these interactions is proposed.

Key words: perseverance, mathematical reasoning, affect, cognition, representation

Introduction

In this small-scale study, I worked with two primary teachers to develop interventions that facilitated children in Year 6 to persevere in mathematical reasoning.

The central importance of reasoning in mathematics learning has been widely argued. For example, Ball and Bass (2003) consider mathematical reasoning to be a basic skill on which children's use of mathematics is founded. However, the development of children's mathematical reasoning is not straightforward; reasoning processes can trace a "zig-zag" route (Lakatos, 1976, p.42) which necessitates on-going cognitive decision making and can involve experiences of becoming and overcoming being "stuck" (Mason, Burton, & Stacey, 2010, p.45). Such mathematical engagement does not occur in isolation from attitudes and emotions; reasoning takes place within an affective context in which there is significant interplay between cognition and affect (Hannula, 2011). In navigating cognitive difficulties a range of emotions can be experienced, not all of which are enabling (Goldin, 2000). Perseverance in mathematical reasoning is required to overcome the cognitive and affective difficulties encountered.

Three key ideas form the theoretical framework for this study; mathematical reasoning, perseverance in mathematical reasoning and affect.

Mathematical reasoning

Mathematical reasoning can be considered to include deductive approaches that lead to formal mathematical proofs and inductive approaches that facilitate the development of knowledge; Polya (1959) broadly interprets these two types as demonstrative and plausible reasoning respectively. In this study, my interpretation of mathematical reasoning was based on Polya's (1959, p.36) "plausible reasoning" and is consistent with Lithner's (2008, p.257) interpretation:

> *Reasoning* is the line of thought adopted to produce assertions and reach conclusions in task solving. It is not necessarily based on formal logic, thus not restricted to proof, and may even be incorrect as long as there are some kinds of sensible (to the reasoner) reasons backing it.

Drawing on the work of Mason et al. (2010) and Stylianides and Stylianides (2006), I identified five key processes in mathematical reasoning: specialising (making trials), spotting patterns and relationships, conjecturing, generalising and convincing.

Perseverance in mathematical reasoning

Lee and Johnston-Wilder (2011, p.1190) identify perseverance as one aspect of the construct mathematical resilience and argue that it is needed to overcome "mathematical difficulties". I have not located a definition of the construct 'perseverance in mathematical reasoning' in literature and hence have sought to articulate this. The nature of the perseverance required to reason in the way that Lithner (2008) describes does not seem to be a dogged, keep trying, repetitive kind of persistence. Rather, it is characterised by the reasoner's movement between reasoning processes whilst overcoming difficulties encountered. For example, rather than persisting and becoming stuck in the process of creating multiple random trials, an initial random trial approach may lead to awareness of pattern, which may then lead to increasingly systematic trials and the formulation of generalisations. Hence, I have articulated the following working definition to describe perseverance in mathematical reasoning: the successful movement between mathematical reasoning processes to create and test assertions and reach and justify conclusions.

Affect

In recent years, there have been significant theoretical developments in the interpretation of the affective domain in relation to learning mathematics (Hannula, 2011; Zan, Brown, Evans & Hannula, 2006). Researchers have begun to apply these to empirical research and this has necessitated the development of appropriate data collection tools. In their exploratory study, Schorr and Goldin (2008, p.135) interpreted students' actions in mathematics lessons by identifying "key affective event[s] […] where significant affect or a significant change in affect […] is expressed or can be inferred". I have drawn on Schorr and Goldin's (2008) study to describe the following key affective indicators: verbal expressions of affect and changes in speed or tone of speech, facial expression and body position.

The study: data presentation and discussion

The study comprised one Baseline Lesson (BL) then four Research Lessons (RL), in each of two classes. Each teacher purposively selected four children from their class based on their assessments of the children's limited perseverance in mathematical reasoning; there were hence eight children in the study group. All the activities in all lessons in the study afforded opportunities for reasoning.

Baseline lesson

In the BL, no intervention was applied. The task involved arranging the numbers 1-5 into a V shape such that each 'arm' of the V totalled the same amount (nrich, 2015a). There was little evidence of mathematical reasoning in the study group. All eight children used a random specialisation approach (Mason et al., 2010) to the task and six were not able to create a successful solution through their random trials. None of the children formed a generalisation or sought to form a convincing argument.

The children expressed puzzlement during exploration and pleasure in creating successful trials. However, there were also instances of negative affect including frustration and despondence. These were evidenced through, for example, a slumped body position and the use of sarcasm in relation to their perceived limited progress.

Research Lessons 1 and 2

The initial intervention in this study facilitated children to use practical mathematical representations in a provisional way. This derived from Papert's (1980) work on the LOGO programming environment in which he utilised the provisional nature of programming to facilitate children to conjecture, make trials and use the resulting data to make improvements. Papert argues that this provisional approach not only facilitates a fallible, trial and improvement approach but also makes children "less intimidated by a fear of being wrong" (Papert, 1980, p.23). In RL1 (Figures 1a, 1b) children were given digit cards and Numicon to manipulate in a provisional way to create an addition pyramid. In RL2 (Figures 1c, 1d) they used Cuisenaire rods in a provisional way to explore creating square ponds, surrounded by square paths.

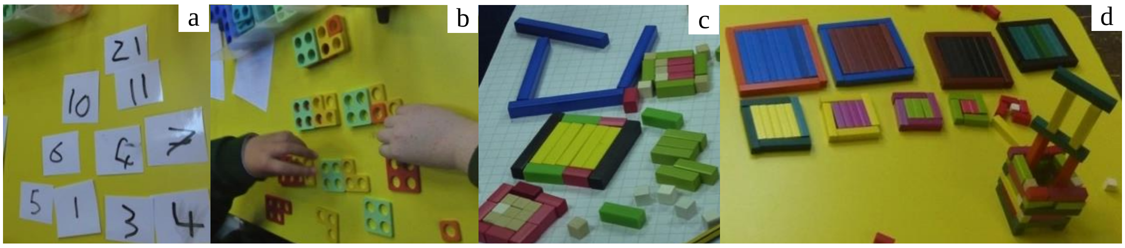


Figure 1: Provisional use of representations in RL1 & RL2

The children's provisional use of representations initially supported them to create random trials. Through this exploration, they became aware of the emerging patterns and relationships. For example, in the post lesson interview, Alice articulated the colour pattern in the paths and ponds as:

> red on the outside, then it turned into the inside. And then the green was on the outside so the green will go on the inside, then purple on the outside then purple on the inside.

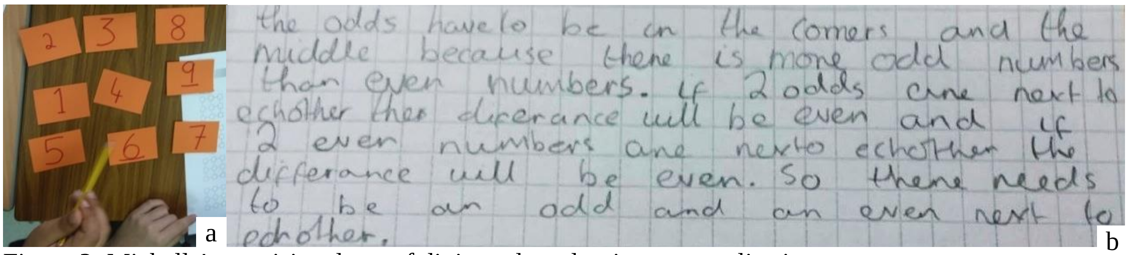
Whilst this led to an increasingly systematic approach to specialisation (Mason et al., 2010), there was very little evidence of the eight children conjecturing, generalising or convincing and hence limited perseverance in mathematical reasoning. However, there was no evidence of the despondency seen in the BL. Rather, there was urgency in the children's speech and actions, pleasure in spotting patterns, frustration when trials were unsuccessful and excitement in establishing a systematic approach; overall, the children reported enjoyment of the activities and the challenge.

Two themes emerged that seemed to contribute to the children's limited perseverance from creating trials and pattern spotting to other reasoning processes. First, there was a need for more time to facilitate systematic approaches and emergence of patterns that might lead to conjecturing and generalising; in both RL1 and RL2 the eight children used much of the lesson time to develop the systematic approaches that led to their increasing awareness of pattern. The second theme was the need for an explicit focus on generalising. Michelle, Alice and Ruby used their awareness of pattern and the relationships between the Cuisenaire rods to construct systematic sequences of examples (Figure 1d) and had achieved this with 25 minutes of the lesson remaining. However, despite the teachers' encouragement to tabulate their findings and seek numerical patterns, they did not do this. Rather, Alice and

Ruby constructed the tower shown in Figure 1d and Michelle sat passively for the remainder of her lesson.

Research Lessons 3 and 4

The teachers and I augmented the intervention such that it facilitated children to use representations in a provisional way, provided additional time, and embedded an explicit focus on generalising and convincing into the task. Hence, RL3 and RL4 took place over two lessons on consecutive days in each class and focused on just one task (Number Differences, nrich, 2015b). This involved arranging the numbers 1-9 into a 3x3 grid such that the difference between adjacent vertical and horizontal numbers was odd. Both teachers asked the children to create a written account of their findings with supporting convincing arguments.



a b

Figure 2: Michelle's provisional use of digit cards and written generalisation

All the children in the study group utilised the time in RL3 and RL4 to make trials. These began with random specialisation, facilitated by their rapid provisional use of the digit cards (Figure 2a), to get a feel for the task and increasingly erred towards a more systematic approach as they developed awareness of the relationships between the numbers. Following this period of exploration, Michelle articulated her generalised findings with convincing arguments (Figure 2b) in two parts. First an empirical generalisation (Mason et al., 2010) in which her argument is anchored (Lithner, 2008) in odd/even number property and second using counter examples:

> The odds have to be in the corners and the middle because there is more odd numbers than even numbers.
>
> If 2 odds are next to ech other the diference will be even and if 2 even numbers are nex to ech other the difference will be even. So there needs to be an odd and an even next to ech other [*sic*].

In the post-lesson interview following RL4, Michelle articulated her feelings about her work in this lesson, relating her sense of pride to her understanding:

> [I am] proud that I know how to do it. I understand it

Michelle's responses in RL3 and RL4 are representative of the study group. All eight children overcame the difficulties that they had previously experienced in persevering, for example, becoming stuck and repeatedly creating random trials or becoming aware of patterns but not using this to conjecture and generalise. All demonstrated movement between mathematical reasoning processes and were able to generalise and form convincing arguments. Similarly, all eight children expressed feelings of pride as well as pleasure in their mathematical reasoning.

Conclusion

The children's provisional use of representation seemed to create affectively enabling conditions that supported them to work with mathematical uncertainty without experiencing the despondency that occurred in the BL. This is perhaps because the provisional approach facilitated exploration and the rapid creation of multiple random trials. The pace of generating data appears to be a significant factor in the children's increasing awareness of the emerging patterns and relationships and their subsequent adoption of a systematic rather than random approach. The intentional, systematic specialisation and noting of patterns and relationships facilitated an enabling affective response and the entire study group reporting enjoyment of the activities and the challenges. However, this initial intervention did not result in the study group persevering in mathematical reasoning such that they developed conjectures, generalisations or convincing arguments. Figure 3 represents the relationships between the initial intervention, the study group's affective response and their mathematical reasoning processes.

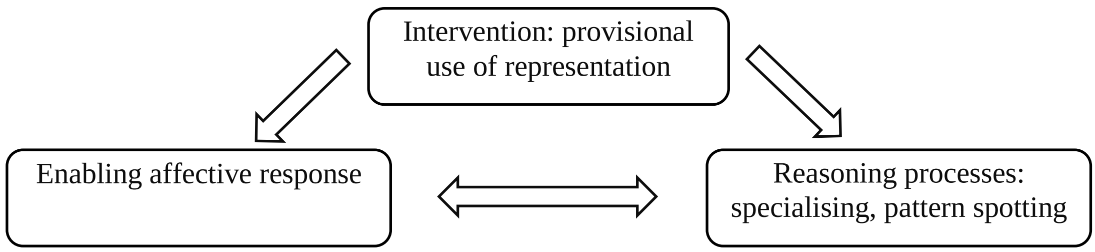


Figure 3: Analytic framework describing the impact of the initial intervention

Following the augmented intervention, the children's provisional use of representations resulted in affective and reasoning responses that were consistent with those described in Figure 3. However, the specific focus on generalising and convincing and additional time to deepen understanding about the emerging patterns and relationships enabled the study group to persevere in their mathematical reasoning: they were able to make trials, form and test conjectures, spot patterns and relationships, generalise their findings and construct arguments to explain these. This also had an affective impact that differed markedly from the enjoyment and pleasure arising from RL1 and RL2; following RL4, the children described feelings of pride and satisfaction and related these to their deeper understanding of the mathematics in the activity. Figure 4 represents the relationships between the augmented intervention, the study group's affective response, their mathematical reasoning processes and their perseverance in mathematical reasoning.

In this study, the explicit focus on generalising and convincing and the time to do this seemed to enable the study group to persevere in mathematical reasoning. This was facilitated by the children's provisional use of representations to create rapid, multiple and increasingly systematic trials from which patterns emerged.

The initial and augmented interventions facilitated productive bi-directional interplay between affect and the cognitive processes of mathematical reasoning.

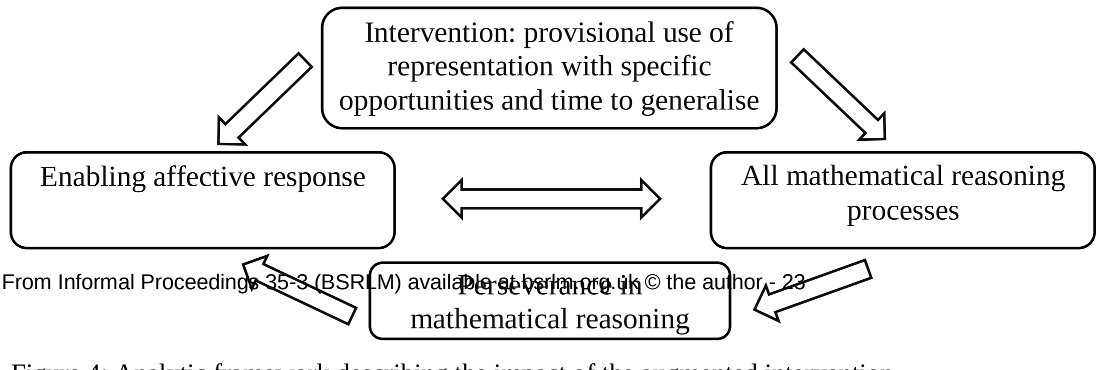


Figure 4: Analytic framework describing the impact of the augmented intervention

However, the children's construction of convincing arguments and the understanding gained from this seemed to be significant in eliciting affective responses of pride and satisfaction. These are valuable outcomes for the children in the study group who had struggled to persevere in mathematical reasoning at the beginning of the research.

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