Making numbers: where we are now

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This paper summarises the initial findings of the Nuffield funded project 'Making numbers', which aims to develop guidance for teachers of children from age three to nine on the use of manipulatives to support the learning of arithmetic. A survey of teachers found they used manipulatives more with younger children and lower attainers of all ages. The literature review considers the history of manipulatives and suggests that a fruitful way of using them is to exploit their ambiguity by relating alternative representations. Key pedagogical factors are also identified.

Keywords: Manipulatives, arithmetic, representation, young children, primary

The project: phase one

This is a Nuffield funded project, running until late 2016. The focus of the project is the use of manipulatives to support arithmetic teaching with children aged three to nine years. The main impetus for the project was the current lack of consensus about manipulative use in England, following the demise of official guidance and the development of virtual resources. The first phase of the project includes a literature review and survey of current practice; the second phase will comprise the development of detailed exemplars of good practice for professional development and use in classrooms, to be disseminated in the third phase.

In terms of arithmetic, our focus is on number sense, which for this age range we summarise as counting, comparison and composition of numbers. These 'three Cs' encapsulate key skills and concepts, including the cardinal counting principle, the relative value of numbers and the ability to flexibly decompose and recompose numbers, which requires understanding of number facts, place value and derived fact strategies. This summary relates to international definitions of number sense, synthesized by Back, Andrews and Sayers (2013) as including the relationship between number and quantity, the meaning of number symbols and vocabulary, ordinal and cardinal aspects of counting, comparisons of number size, different representations, simple arithmetic operations and number patterns, including recognizing missing numbers. This analysis suggests that manipulatives may have diverse roles in supporting different aspects of number sense.

Our working definition of manipulatives is 'objects that can be handled and moved and are used to develop learners' understanding of a mathematical situation'. They may be everyday objects, child- made or commercially manufactured. Due to the constraints of this project, virtual manipulatives are not included, although we recognize their advantages, their integrated use with 3D manipulatives and the growing literature on their effectiveness. Our initial assumptions are that manipulatives might be effective in supporting arithmetic learning to help children make sense of arithmetic, to increase engagement, to develop visual images and understanding, to develop representational repertoires, as part of an enquiry-based community and as tools to solve problems, to investigate patterns and relationships and to demonstrate results and reasoning.

We have initially classified manipulatives according to their degree of structure and whether they are designed for teaching, for instance categorising everyday objects as unstructured, but dice, dominoes and money as structured, and categorising interlocking cubes as designed for teaching but unstructured, as opposed to more structured bead strings, 10-frames, Numicon (ref), Dienes' apparatus, Montessori golden beads, Cuisenaire rods, Hungarian number pictures, abacus, place value cards, and place value counters. We have identified some commonly used 2D resources as non- manipulatives, including numbered and empty number lines and 100 squares, although they may be used interactively and with manipulatives.

Survey of manipulative use

In the first phase of the project we have conducted a questionnaire survey of manipulative use with 457 teachers in 35 schools, including a mixture of urban, suburban and rural settings in the midlands and south east England. This was a convenience sample of schools we were already working with, and as such was likely to represent schools positively engaged in developing mathematics teaching. The questionnaire asked teachers of all age groups from three to eleven years which manipulatives they had used to teach number in the preceding four weeks, offering a list of uncategorised examples. The results indicated that, of structured manipulatives, the ones most teachers used were dice (70%), Numicon (50%) Dienes' base ten apparatus (42%) and coins and notes (41%) followed by place value cards and beadstrings (both 36%). With hindsight we might have included fingers and distinguished dice with or without numerals. With unstructured apparatus, counters and interlocking cubes were most common. Of the 'non- manipulative' resources, 100 squares were used most (69%), followed by numbered lines and tracks (57%), then empty number lines (49%). Perhaps surprisingly, even teachers of the oldest children used 100 squares more than number lines.

We also asked teachers of all ages, including the Early Years Foundation Stage (EYFS) whether they used manipulatives with lower, average or higher attainers (defined by quartiles of the highest and lowest achievers). This showed a greater use of manipulatives with younger children and that teachers of the oldest children were twice as likely to use manipulatives with lower than with higher attainers.

Overall use of manipulatives (in percentages)				
Phase	**Lower attainers**	**Middle attainers**	**Higher attainers**	**Number of classes**
EYFS	98	98	95	87
Key Stage 1	98	95	87	152
Lower Key Stage 2	93	80	52	114
Upper Key Stage 2	92	70	45	131
TOTAL	**94**	**85**	**68**	**484**

These results will be further investigated in focus group discussions in the next phase of the project.

The history of manipulative use

The first phase of the project has also included a literature review, considering the history and theory of manipulatives use, as well as impact studies. Manipulatives to support arithmetic have a long history worldwide, starting with pebbles and abacuses. Structured educational materials began in the early 19th century with wooden rods representing numbers to 12 devised by Froebel as part of his set of building blocks and many resources for teaching arithmetic, including base ten apparatus, odd and even number images and overlapping place value cards were developed early in the 20th century by practising teachers, such as Montessori, Cuisenaire and Stern. The use of different resources waxed and waned in Europe and North America, influenced by prevailing theories. For instance, Montessori's approach was considered too structured by the USA progressives Dewey and Kilpatrick (Gutek, 2004). However, the Piagetian theory of a concrete operations stage was influential in endorsing the use of manipulatives with younger children. In the 1960s the 'new mathematics', of which Dienes was one of the advocates, encouraged number investigations with structured apparatus. Advances in the manufacture of plastics also created developments, from Stern's use of Bakelite to the invention of interlocking cubes.

In the 1970s in the Netherlands, Realistic Mathematics Education (RME) emphasized contextualized problem solving with diagrammatic models (Streefland 1991). In the 1980s and 1990s, mathematics education researchers, including members of the Psychology of Mathematics Education (PME) Working Group on Representations became concerned with children's difficulties in understanding mathematical representations (Goldin, 1998). For instance, information processing theory was used to identify the complex cognitive mappings involved in using base ten materials to support column addition and subtraction algorithms. Interestingly, such critiques tend to focus on manipulatives used to teach calculation rather than to investigate numerical structures. RME researchers analysed children's actual and unintended use of the abacus to support calculation and developed new manipulatives such as beadstrings and the counting rack to capitalize on children's intuitive 'shortcuts' and avoid reliance on counting (Gravemeijer,1991). In England, concerns about children's calculation skills resulted in the government's numeracy projects drawing on Dutch research and recommending beadstrings as precursors to empty number lines (Department for Education (DfE) 2010). Gradually practical apparatus has been less used, partly due to the prevalence of interactive whiteboards (Brown 2014). More recently, English government interest in the approaches of high performing jurisdictions has resulted in the use of Singapore textbooks advocating a Brunerian concrete- pictorial- abstract approach. This includes the 'bar model', with the use of cubes or Cuisenaire rods, as recommended by the National Centre for Excellence in Teaching Mathematics (NCETM). Recently Stern's odd and even number plates have been successfully promoted as Numicon. In summary, ideological approaches, technological advances and government policies, as well as psychological analyses of children's learning, have resulted in the changing use and cycles of rediscovery of manipulatives to teach arithmetic.

Theories of representation

Whereas the early developers of instructional manipulatives considered their representation of number relationships as transparent, more recently the processes involved in deriving mathematical understanding from physical objects have been problematized. In summary, the main arguments from theorists are that:

- all representations are necessarily partial, and can only represent some aspects of mathematical structures (Vergnaud, 1987)
- different modes of mental or internal representations may be imagistic (including visual-spatial, tactile-kinasthetic and auditory–rhythmic), verbal, written or affective (Goldin, 2002). This implies that we should consider imagined movements, written calculations and emotions as well as words and images as internally constructed mathematical representations.
- the nature of understanding in the brain is a network of representations in different modes (Goswami and Bryant, 2007). Understanding is not about condensing associations into an abstract concept which is represented symbolically and it is the complexity of the network which indicates deeper or more sophisticated understanding
- making connections between external representations makes connections between internal representations (Hiebert and Carpenter 1992). This implies that representing the same idea in different ways, discussing differences between representations, or just putting into words what is shown by an arrangement of materials, will help build networks of understanding.

Theories also raise some major questions:

- How do particular manipulatives represent mathematical relations, by emphasising or ignoring aspects? For instance, Cuisenaire rods in order show numbers increasing by one, whereas Numicon emphasises odd and even numbers and doubles.
- How do individual children make sense of the manipulatives? Are some representations too opaque or complex for some children? For instance younger children may have difficulties in understanding objects symbolically and Harries and Barmby (2006) report older children’s difficulties in understanding arrays multiplicatively.
- How does children’s understanding of relationships move from concrete to abstract? It is not clear what we mean by children working with abstract concepts and how this might develop from working with manipulatives.
- Are there missing manipulatives which yet remain to be invented, based on children’s ways of thinking?

The implications from theory are that, while manipulatives have potential benefits in terms of tactile, muscular and affective memory modes, we need to consider carefully how children make sense of them. There are potentially different roles for manipulatives in investigating mathematical relationships and in modelling problems set in everyday contexts. It seems that much may be gained by exploiting the ambiguity and partiality of representations and by children expressing the relations they see in different ways, with other children and for themselves. One advantage of manipulatives is that they can be easily moved and rearranged to support reasoning in discussion. We also need to consider how children might use talk, drawings and writing to record their mathematical activity and give insights into these processes.

Pedagogical implications

Empirical studies of the effectiveness of manipulatives tend to be inconclusive, as meta-analyses show (e.g. Carbonneau, Marley & Selig, 2013), perhaps because they are unable to control for the many variables involved: however they clearly indicate the importance of teaching processes. Studies emphasise teachers' understanding of the role of manipulatives linked to the mathematics being taught. For instance, Moscardini (2009) identifies teachers either using manipulatives as tools to help children make sense of problems or as crutches to enable them to complete a procedure (often poorly understood). In summary, studies point to several key factors contributing to the effective use of manipulatives. Firstly, teachers need to carefully identify the prerequisite understanding for the mathematics they intend children to learn and to assess children accordingly: for instance, manipulatives will not help children understand decomposition if they do not first understand two-digit subtraction or place value. Materials need to be carefully matched to the mathematical relations involved. Children need to be familiar with the materials, and to have satisfied their curiosity by playing with them, before being directed to specific activities. Particular manipulatives have their own protocols for use, for instance in how they may be arranged, and the vocabulary used, which children need to learn. It seems that the most productive activities are open–ended challenges like, 'How many can you find which are equivalent to this?' which can provide varied examples and enable generalising, for instance about numbers with the same total or in the same proportions. Encouraging children to talk about what they are doing and justify their reasoning is likely to help them make connections. Removing the manipulatives and requiring children to visualize then helps them internalise representations. Encouraging children's own recording has been shown to develop links with abstract symbols and generalisation to smaller and bigger numbers, as well as giving teachers insights to children's understanding and thinking processes. Finally, for children to engage in the kind of exploratory activity which encourages generalising, an inclusive, enquiry-based classroom community is required, where high expectations for all and safe risk-taking help children develop positive identities as mathematics learners. All of these aspects are exemplified in the work of Madeleine Goutard (1964), with Cuisenaire in Canada, where she developed impressive abstract generalisations in the 'free' mathematical writing of seven year olds, who produced equivalent statements like $10 = (1/2 \times 2) \times (2/2 \times 10) = 10^1$. Goutard recommended free play followed by open-ended challenges, such as finding families with the same difference or equivalent fractions, developing Gattegno's approach. She strongly advocated valuing all children's responses, then analyzing and building on them, with appropriate challenges, encouraging children to discuss and justify their views. Clearly Goutard also was very knowledgeable about the possible progression of children's mathematical thinking, and had extremely high expectations. In summary, it therefore seems that some major issues for teachers are:

- how to select and use manipulatives- are they tools or crutches?
- how to select activities which use manipulatives effectively
- how to support the transition between practical activity and symbolic recording.

The next phase of the project will seek to uncover and develop exemplars of good practice. For instance, current work with Cuisenaire, as shown by Ainsworth (2013) makes children's mathematical learning transparent in their exploration of relationships, finding commonalities in examples and generalising. In future work we intend particularly to explore children's own recording and to build on the interests of younger children.

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