Gaining meaning for expressions with Grid Algebra: developing the CAPS framework

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Through examples from the early stages of a study with Grade 7 low-attaining learners from Malta using Grid Algebra, we argue for the significance of actions, pictures and symbols in learners developing concepts for formal expressions. The interrelationship between these forms the basis of the CAPS (Concept, Action, Picture and Symbol) framework as an analytical tool, which is presented in this paper.

Keywords: early algebra; notation; arithmetic; expressions; procept

Notation interpretations and representations

Learners' difficulties with number and algebra can stem from their misinterpretation of mathematical symbols and their properties. For instance, even at the start of secondary school, some children may have difficulty knowing which of the four arithmetic operations are commutative (Booth, 1988; MacGregor, 1996). Some research revealed learners' lack of knowledge about the inverse property of operations (Gallardo & Rojano, 1987) and their failure to perceive cancellation (Herscovics & Linchevski, 1994).

Other research focused on issues regarding learners' interpretations of expressions. For example, while some learners where found to pick up on visual cues, such as spaces, in expressions to construct correct meanings for expressions (Kirshner, 1989), others were found to struggle with some problems due to their interpretation of visual cues. Borg (1997) found that some learners got confused when they wanted to make *n* the subject of the formula $r = (n + mc)^2$ just because they interpreted the brackets as a signal to expand the expression on the right. Herscovics and Linchevski (1994) reported about learners who interpreted the lack of space or symbol between 3 and n in the expression $3n$ as simply the number 3 concatenated with the letter n, where a substitution of $n = 2$ meant the replacement of n with 2, thus obtaining 32 instead of 3(2) or 3×2. The left-to-right writing of some expressions coupled with the convention of left-to-right reading of text seems to be a visual cue for some learners to work out the operations of any expression in left-to-right order (for example, Blando, Kelly, Schneider, & Sleeman, 1989).

The learning of new formal notation proves to be challenging for many learners (Van Amerom, 2003). This includes extending meanings for notations already learnt, such as those associated with the divisor line of a fraction which some learners find confusing when they first encounter it as signifying division of the numerator by the denominator (Hewitt, 2009). Here learners need to accommodate the notion that the expression $\frac{a}{b}$ is not just a fraction but also a new notation signifying a÷b. Then again, learners may need to regard the expression $\frac{a}{b}$ not only as a process of dividing a by b but also as an object in its own right which may be operated on to form new expressions like $\frac{a}{b}+c$.

The conceptual reconstruction of an expression resulting from a process into a mathematical entity is well documented in the literature and the notion may find its origin in Piaget's (1985, p.49) contention that "actions or operations become thematized objects of thought". Learners' encapsulation (Ayers, Davis, Dubinsky, & Lewin, 1988), reification (Sfard & Linchevski, 1994), integration operation (Steffe & Cobb, 1988), or entitication (Harel & Kaput, 1991) of a process into a conceptual entity (Greeno, 1983) enables them to conceptualise a string of mathematical symbols as both a process and an object, or what Tall (1991) called procept.

The interplay between what learners visualise and what they conceptualise is an interesting, yet complex issue for all teachers. Vergnaud (1987) and later Kaput (1991) used the terms signifier and signified in the context of mathematics education to denote respectively external representations and internal interpretations of mathematical symbols. They borrowed these terms from Saussure (1966) who was the first to come up with this dichotomous model of signs: the interpreted and the represented. This model, however, seems to disregard the diverse representations which emanate from and inform mental structures and the interconnection between those representations. In particular, Bruner (1966) identified three important representations: the enactive (resulting from learners' actions), the iconic (resulting from learners' drawings) and the symbolic (resulting from learners' signs, symbols, or notations), each of which play important roles in the formation and manifestation of concepts and, from the side of the teacher, in the development of hypotheses about learners' understandings (Kaput, 1991).

The study we are discussing here looked into learners' interactions with standard mathematical notation which appeared within the software Grid Algebra and which was either new or carried an extended meaning for the learners. Examples offered here were at a time before a letter was used within any expression and our focus for this paper is on arguing for the relevance of an analytical framework which focuses on the relationship between concepts, actions, pictures and symbols (CAPS).

The school, the pupils, the software, and the method

The study took place in a Maltese secondary school in which learners were set according to their achievement levels in Mathematics, English, and Maltese. Learners entering Grade 7 (aged 11-12) were set according to the grade they obtained in a national benchmark examination at the end of Grade 6.

The pupils in the study were in the lowest of three achievement sets where their Grade 6 benchmark scores varied from 1.04 to 3.18 standard deviations below the mean. With the exception of Tony (all names are pseudonyms), they had some kind of condition which could hinder their learning and achievement. Jordan had a speech/language difficulty, Dan had ADHD and literacy problems, Omar had dyslexia and literacy problems, Joseph had ADHD and slight dyslexia and Dwayne suffered from coeliac disease. However, with the exception of some difficulties in understanding long questions and some minor behavioural issues in the classroom it seemed that these children were not hindered in their mathematical activities due to their conditions. In fact, for the lessons in which data was collected, the learning support assistants assigned to facilitate their learning were not present.

The software used in the study was Grid Algebra which has an interface showing a grid based on the multiplication tables, with the one times table in row 1, two times table in row 2, etc. Numbers or letters can be entered in cells according to the designated tables and their relationship with any existing numbers appearing in the grid. Cells and their contents can be moved around the grid and the consequences of those moves are shown in an expression appearing in the appropriate cell (see Figure 1 for an example of some movements). The cell with a new expression can be dragged again to a new destination where a new expression will be formed.

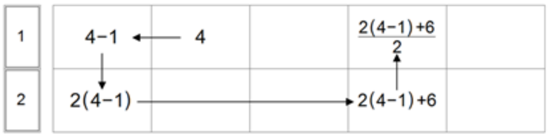

Figure 1: An example of moving a number around the grid which results in expressions being created.

Qualitative data was collected from videos of the lessons and semi-structured interviews in which the learners were required to answer mathematical questions in writing and give reasons for their answers. Other data was collected from videos of computer screen activities of the children and from written work. The whole study was spread over the whole scholastic year, with twenty double lessons (1600 min) and five interviews for each learner. However, as stated earlier, the examples used here were at the early stages and this paper does not focus on results of the study but on the relevance for CAPS framework as an analytical tool.

Action

In the early stages the grid was used as a static image with challenges for learners to work out which number should be in a certain cell, given a number in a different cell. In this way the grid was a *picture* with which the learners worked. In the situation in Figure 2, Omar had highlighted one cell and was helping Jordan work out which number should be in that cell.

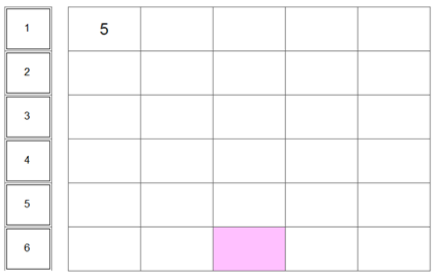

Figure 2: the task of finding out which number should go into the highlighted cell.

Omar made skipping movements with his hand going from the 5 cell to the right (to cells which would be 6 and 7) and then when moving down to the highlighted cell (going through cells which would have the numbers 14, 21, 28, 35 and finally 42). These physical actions on the grid relate the number 42 (which they put afterwards into the highlighted cell) to other numbers within the grid. We argue that their developing concept of number is made through seeing relationships between numbers and not just as a number in its own right or as merely a counting number, such as 42 coming after 41. Even with a static picture, the actions were significant in the learners' development of relationships between numbers. Later on, once the learners explored dragging numbers and expressions around the grid, actions became a key tool in creating expressions and carrying out many of the Grid Algebra tasks.

Symbol

Initially the learners were not familiar with the particular use of certain mathematical symbols, such as the division line and the use of brackets. For example, in an interview before the learners had met Grid Algebra, Jordan was asked about $10(5 + 2)$

and knew that he had to work out 5 + 2 first but had never met a number before a bracket without an operator in-between. He felt the answer was 17 and in saying so he used an action bringing his hands together as if to squash the 10 and the 7 (from 5+2).

Work with Grid Algebra in dragging expressions around the grid, shifted focus onto an expression as an object as well as a process to be carried out. One activity had a number in one cell with a different cell highlighted where the learners had to decide what expression could result from the number being dragged into that cell. A dialogue between Dwayne and Tony, for example, included talk where they referred to $10+3$ as 13 and later on talking about $2(13\times 3)$ as "two times thirteen times three". So there was a mix of seeing an expression as a process and seeing it as an object.

As well as the gradual shift to seeing expressions as objects, the new notation of a number appearing in front of a bracketed expression was now seen as multiplication just as much as the multiplication sign.

Picture

We have already indicated that the picture of the grid was significant for learners in seeing relationships between numbers and in their acceptance of some of the new symbolism. In addition, each expression also began to have its own associated picture. For example, one of the computer generated tasks gives a number in one cell and an expression in another cell which is the result of dragging the original number around the grid. The task is to identify what that journey was by clicking on the correct sequence of intermediate cells from the start number to the final expression (see Figure 3).

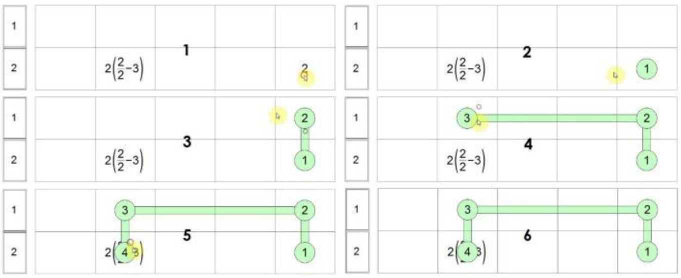

Figure 3: A sequence of screen shots with Dan and Joseph finding the journey associated with the given expression (the numbers 1-6 show the sequence of screen shots)

The expression $2\left(\frac{2}{2}-3\right)$ can be viewed as an historical artefact resulting from a journey made from the start number 2. Thus, as well as representing a series of arithmetic operations, it tells the story of a physical journey. Figure 3 shows Dan and Joseph gradually building up the associated picture for that expression in terms of the journey made to create that expression. The focus is now on the expression as an object. For example, the expression $2-6$ could also appear in the same cell as it is numerically equivalent, however it would represent a completely different journey. As such the evaluation of an expression, or the transformation of an expression into an equivalent expression, would destroy the detail of the journey taken. The focus is now on the expression as an object, not to be manipulated into an equivalent expression or any part evaluated. This association of an expression with the picture of a journey was continued on paper (see, for example, Jordan's drawing of the journey for a more complex expression in Figure 4).

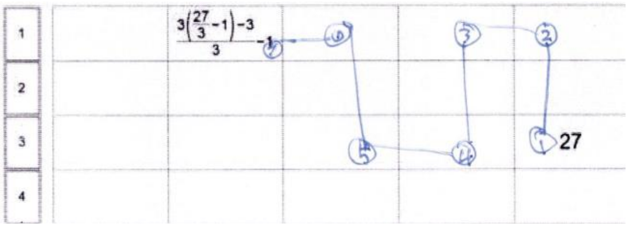

Figure 4: Jordan's drawing of the journey associated with the given expression.

Concept and the CAPS framework

Concepts are being built up through a mix of the actions, pictures and symbols which form the work with Grid Algebra. Here we briefly reflect back upon the last three sections whilst offering a diagrammatic image of the CAPS framework in Figure 5. Grid Algebra affords the carrying out of actions, either virtually or physically, within the grid. The grid itself offers a picture which relates numbers to each other and this is reinforced through movements which take place in the grid. Mercer (2000, p.67) talks about how words gather meaning from the "company they keep" and we argue that the same applies to numbers as they gather meaning from their relationship with other numbers. Movement within the grid stresses such relationships with the software offering the notational consequence of these movements. Thus, a movement for multiplication can result in the software showing the use of brackets with a number immediately preceding it. Since the only thing which appears in the cells of the grid is notation, the learners become used to the way operations are notated whilst they focus on the various Grid Algebra tasks. Thus their concept of the symbolic notation is connected with the action of dragging in the grid.

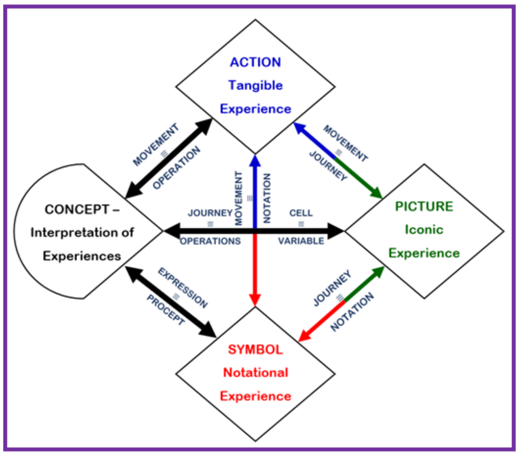

Figure 5: the CAPS framework showing the relationship between different elements when using Grid Algebra.

A symbolic expression not only represents a series of mathematical operations but also a picture of a physical journey carried out on the grid. Such a conceptual way of viewing an expression places emphasis on the expression as an object and not just a process to be carried out. This helps learners gain a proceptual (Tall, 1991) way of working with expressions.

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