The role of prospective teachers' mathematical knowledge in recognising students' understanding: the process of unitising

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This study analyses the role of prospective teachers' mathematical knowledge in recognising students' understanding of the idea of unitising process (as a component of proportional reasoning). 92 prospective primary teachers analysed primary students' answers to 12 school problems about different components of proportional reasoning. To each problem prospective teachers analysed three answers that showed different characteristics of the development of proportional reasoning. The focus of this paper is on the problem that could be solved by the unitising process. Prospective teachers had to answer 4 questions: a) about mathematics elements in the school problems; b) about the recognition of students' understanding; c) and d): about how to modify the school problem to support the student's conceptual understanding. Results suggest that the way in which prospective teachers understood the unitising process influence on (i) what they considered to be the learning objective, (ii) what they recognised as evidence of the primary students' understanding and (iii) how they modified the school problems to help the primary students to develop the understanding of the unitising process.

Keywords: professional noticing; students' understanding; unitising; proportional reasoning; teacher's knowledge

Introduction

The development of proportional reasoning is an important topic in the Primary and Secondary school curricula. However, previous studies have shown primary and secondary school students' difficulties in this development. In fact, recent studies have shown that the teaching and learning of the ideas of ratio and proportion involved in the development of proportional reasoning is not an easy task for teachers (Livy & Vale, 2011). The teacher's knowledge needed to teach these concepts has an important role in the development of teachers' professional competences related to the recognition of students' mathematical understanding (the skill of noticing students' mathematical thinking).

The research presented here is embedded in two lines of research: studies focused on the teachers' skill of noticing students' mathematical thinking and, studies focused on the development of proportional reasoning.

Theoretical framework

The skill of noticing students' mathematical thinking

Recent research indicates that being able to identify relevant aspects of teaching and learning situations and interpret them to take instructional decisions (Mason, 2002; Jacobs, Lamb & Philipp 2010) is an important teaching skill (professional noticing). These studies have also provided contexts for the development of this skill in teacher education programs. A particular focus is the skill of noticing students' mathematical thinking (Jacobs et al., 2010), understood as recognising evidence of the student understanding to take instructional decisions.

Previous research has shown that identifying the relevant mathematical elements of the problems (mathematical knowledge) plays an important role in recognising evidence of students' mathematical understanding (Bartell, Webel, Bowen & Dyson, 2013; Fernández, Llinares & Valls, 2012; Sánchez-Matamoros, Fernández & Llinares, 2014). Therefore, the identification by prospective teachers of the key mathematical concepts (key developmental understanding (KDU), Simon, 2006) plays an important role in recognising the characteristics of students' understanding and also in taking instructional decisions.

The development of proportional reasoning

According to Lamon (2007), proportional reasoning integrates different components: the meanings of the mathematical concepts (rational number interpretations: ratio, part-whole, measure, quotient and operator) and the ways of reasoning with these mathematical concepts (unitising, up and down, relational thinking, and covariance). Pitta-Pantazi and Christou (2011) based on Lamon's characterisation, added the ability to solve missing-value proportional problems and the ability to discriminate proportional and non-proportional situations.

We are going to focus in one of these components considered as a KDU in the learning of proportional reasoning: the unitising process. For Lamon (2007, p.630), "**unitising** is the cognitive process of mentally chunking or restructuring a given quantity into familiar or manageable or conveniently sized pieces to operate with that quantity". This process involves the construction of a reference unit from the relationship between the amounts and uses this new unit to compare or operate. In the following example "The cereal box with 16kg (A) costs 3.36€ and the cereal box with 12kg (B) costs 2.64€. Which cereal box is cheaper?" we can take as a reference unit to compare, the cost of 1kg, the cost of 4kg, the cost of 12kg… For instance, if we calculate how much A cost if it was 12kg, we obtain that it would cost 2.52€, therefore the first box is cheaper.

The unitising process can be considered as a key element in the development of proportional reasoning. Taking into account these aspects, our research questions are: (1) How do prospective teachers understand the unitising process and how does prospective teachers' understanding influence the recognition of students' understanding? and (2) Which instructional decisions do prospective teachers take to support the students' conceptual development?

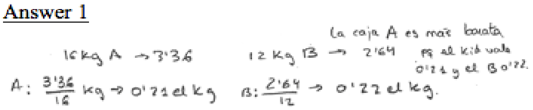
Participants and the task

The participants were 92 prospective primary teachers who studied the third year of the degree to become a primary school teacher at the University of Alicante (Spain). They had attended a course focused on numerical sense (first year) and one focused on geometrical sense (second year). In the third year, they attended a course about teaching and learning of mathematics in primary school and one of the units was about proportional reasoning. Data were collected after this unit.

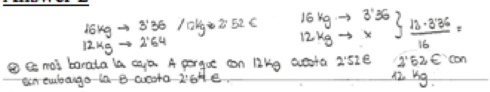
Prospective teachers solved a task consisting of 12 primary school problems related to the 12 components of proportional reasoning and three primary school students' answers to each problem that showed different characteristics of students' understanding of each component. Prospective teachers had to answer four questions related to (a) the learning objective of the primary school problem; (b) the recognition of students' mathematical understanding and; (c) and (d) the instructional decisions prospective teachers take to respond on the basis of students' understanding. In Figure 1, the primary school problem related to the unitising process component, the three primary school students' answers and the four questions are shown.

The cereals box with 16kg (A) costs 3.36€ and the cereals box with 12kg (B) costs 2.64€. Which cereals box is cheaper?

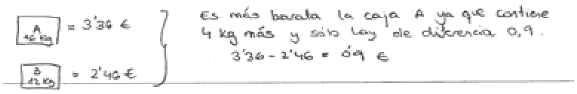
Answer 1



Answer 2



Answer 3



a) What mathematical concepts must a primary school student know to solve this task? Justify your answer.
b) How does the understanding of mathematical concepts involved in each of the students' answers manifest? Justify your answer.
c) If a student does not understand the mathematical concepts involved, how would you change the task to help the students to understand these concepts? Justify your answer.
d) If a student understands the mathematical concepts involved, how would you change the task to increase the students' understanding of the mathematical concepts involved? Justify your answer.

Figure 1. Prospective teachers' task related to the unitising process.

To solve this problem, primary school students have to look for a reference unit that allows them to compare both cereal boxes. In this case, students could use the cost of 1kg, the cost of 4 kg…The three students' answers were selected taking into account different students' strategies that display different understanding. In answer 1, the student identifies the cost of 1 kg and then compares them. In answer 2, the student identifies as a unit the cost of '12 kg' and then compares the two boxes. In both answers, the students restructure a given quantity into a conveniently sized unit to compare. In answer 3, the student identifies the difference between the prizes without taking into account the kg of each box (the student identifies an additive relationship instead of a multiplicative relationship).

Analysis

Data are prospective teachers' answers to the four questions of the task. Three researchers categorised them analysing the answers to each question individually. Validity and reliability were established by comparing sets of independent results, citing specific examples, clarifying the coding schemes and re-coding the data until 100% of agreement was achieved.

With regard to the mathematical content that prospective teachers considered was involved in the primary school problem (question a), we identified two categories: prospective teachers who identified the unitising process (that is, prospective teachers who identified the idea of restructuring a given quantity into familiar or manageable or conveniently sized pieces (or unit) to compare), and prospective teachers who identified other ideas not related to the unitising process. We jointly analysed question (a) and (b) in order to check if, although prospective teacher had not written the mathematical content of the task in question (a), they had used it to recognise students' understanding (question b).

We identified three categories of prospective teachers answers related to how they recognised students' understanding (question b; table 1).

Table 1. Categories and examples of how prospective teachers recognised students' understanding

Prospective teachers who provided general comments based on the correctness of the answer	"*Answer 1: the student solves the task correctly because he knows the concepts involved. Answer 2: the student solves the task correctly because he knows the concepts involved. Answer 3: the student doesn't solve the task correctly because he doesn't use the concepts involved, he uses an additive strategy"*	This prospective teacher based his justification on whether the answer was right or wrong
Prospective teachers who based their comments on a simple description of the answer	*"In answer 1, the student uses the external ratios to solve the problem. In answer 2, the student looks for the price of 12 kg. The answer 3 is not correct because the student applies a subtraction"*	This prospective teacher only described what the student did
Prospective teachers who recognised students' understanding and provided evidence of this understanding	"*In answer 1, he identifies the ratio €/kg and then compares both ratios. In answer 2, he uses 12 kg as a unit to compare and does a rule of three to know how much are 12 kg. In answer 3, he doesn't use ratios, he solves the task with an additive strategy*"	This prospective teacher identified that the first student used 1kg (reference unit) to compare, the second student identified 12 kg as the unit to compare, and the third student used an incorrect additive strategy

With respect to the instructional decisions given by prospective teachers (questions c and d), we will show the categories identified in the results section.

Results

We identified several categories of prospective teachers' answers depending on whether or not they had identified the unitising process as the key mathematical content and on whether they provided general comments, described students' answers or recognised evidence of students understanding (Table 2).

Table 2. Categories of prospective teachers' answers

Identify the unitising process as KDU	Recognise students' understanding (UI)	16
	Describe students' answers (UD)	20
	Provide general comments (UG)	6
Do not identify the unitising process as KDU	Describe students' answers (OD)	24
	Provide general comments (OG)	26

Results show that only prospective teachers who were able to recognise evidence of students' understanding had identified the unitising process as the key mathematical content involved (UI). For these prospective teachers, the fact of identifying the unitising process as a KDU led them to recognise evidence of students' understanding. However, there were prospective teachers who had identified the key mathematical concept but they only described students' answers (UD) or provided general comments about the students' understandings (based on the correctness, UG). It is possible that these prospective teachers had the mathematical content knowledge but they did not know how to use this content to recognise students' understanding (that it is part of the mathematical knowledge for teaching). Prospective teachers who did not identify the unitising process as the KDU only described the students' answers or provided general comments (OD and OG).

On the other hand, Table 3 shows the characteristics of the problems that prospective teachers took into account in order to modify the school problem to help students who do not understand the concept.

Table 3. Characteristics of the problems that prospective teachers took into account to modify the school problem to help students who do not understand the concept

	UI	UD	UG	OD	OG
Integer numbers // smaller numbers	5	12	4	11	9
Integer ratios	8	7	2	7	6
Difference between ratios more significant	0	2	0	0	0
Comparison of one magnitude	2	2	0	3	1
Explain the content	4	1	1	4	4
Blank answer // Nonsense answer	2	2	0	6	10

Results show that the number of nonsense answers or blank answers increased in prospective teachers who had not identified the key mathematical content (unitising process) and they only described students' answers or gave general comments (OD and OG). Furthermore, prospective teachers who had identified the key mathematical content took into account more characteristics of the problems to modify the school problem. Table 4 presents the characteristics of the problems that prospective teachers took into account in order to modify the school problem to improve students' understanding.

Table 4. Characteristics of the problems that prospective teachers took into account to modify the school problem to improve students' understanding

	UI	UD	UG	OD	OG
Similar ratios	0	2	1	1	1
Bigger numbers // rational numbers	5	16	1	5	9
More elements to compare	2	0	1	2	0
Use different strategies	0	2	0	1	0
Change the context	2	1	0	3	1
Blank answer // nonsense answer	7	11	3	12	17

There were more prospective teachers who proposed nonsense answers or blank answers in the question focused on modifying the school problem to improve students' understanding than in the question to help students who do not understand the concept. These data suggest that it was more difficult for prospective teachers to modify the school problem to improve students' understanding than to help students who do not understand the concept. Furthermore, the number of nonsense answers or blank answers increased in prospective teachers who had not identified the key mathematical content and they only described students' answers or provided general comments.

Conclusions and discussion

Identifying the unitising process as a KDU of the learning of proportional reasoning led prospective teachers to recognise students' understanding and provide more variety in problem modification to help students who did not understand the concept or to improve students' understanding. However, although some prospective teachers had recognised students' understanding, they had difficulties in providing a school problem modification (particularly, the modification of the school problem related to the improvement of students' understanding).

On the other hand, there were prospective teachers who had identified the key mathematical content but they described students' answers or gave general comments about students' understanding. This result could indicate that prospective teachers had the mathematical content knowledge but they had difficulties in recognising evidence of students' understanding (that it is part of the mathematical knowledge for teaching). On the other hand, the fact that some prospective teachers gave general comments even though they had identified the mathematical content could indicate that these prospective teachers have the belief that a student answer could just be 'right or wrong' (dualism; Copes, 1982)

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