Comparison of Students' Understanding of Functions throughout School Years in Israel and England

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Little is known about the overall growth of students' understanding of functions throughout schooling. We have been identifying the development of students' understanding of concepts which contribute to understanding functions throughout school in two different curriculum systems: in the UK and in Israel. In this paper we shall present some of the comparative findings and make conjectures about differences.

Keywords: functions; variables; covariation; sequences

Introduction

In Ayalon, Lerman & Watson (2013) we described the purposes and design process for a survey instrument intended to be used in two countries, Israel and England. The aim is to learn more about how concepts relevant for understanding functions develop, using comparative data from two different curricula systems. In this paper we present brief details of five of the six tasks in the survey, and conjectures about the reasons for different performances by students in the two countries. There are several distinct routes for development of functions through school: generalisation of sequences; graphical representation of realistic data; sets of points generated from equations/formulae; input/output models; relations and covariation between variables; expressions for mappings between sets (Leinhardt, Zaslavsky & Stein, 1990). All these could have the word 'function' attached. Our survey provided tasks which could reveal any or all of these ideas. We found the following features to be important in distinguishing between students' responses: identification of discrete and continuous variables, and simple or compound variables; identifying relations between variables; recognising covariation. A key distinction was between correspondence approaches to generalising functions, i.e. expressing them as input/output formulae - expressions that connect data pairs, and covariation approaches, i.e. expressing them as relative changes in two variables, taking a starting value into account.

Implementation of the study

Participating schools provided data from highest achieving classes; these were comparable as a proportion of school cohorts in each of their contributing years. (We also have data for other classes, not reported in this paper.) We use random anonymised samples of 10 scripts from each class. In this way we received, from such classes, 70 scripts from UK years 7 to 13 inclusive, and 60 scripts from Israel years 7 to 12 (these would be equivalent to years 8 to 13 in UK). In comparing Israel and English students' responses we are not taking precise age into account, nor years in school. Rather we are looking at features of progression in concepts in both countries, and comparing progressions to see what can be said to be common and what can be said to be different. We also look for differences in prevalence of particular approaches. Progression in conceptual understanding depends on curriculum order and classroom culture as well as students' learning and, possibly, maturation. If a concept is taught in a particular year and not mentioned again for a couple of years we would expect to see some falling off in its use in the survey, maybe reappearing later when it reappears in the curriculum. If a concept is taught in a particular year and subsequent teaching and classroom culture refer to it and use it we would expect to see sustained use in the survey, possibly with some increase as the students' concept image becomes more complex and embedded. For example, a teacher might continue to use a range of function representations, language, and formal notation in areas of mathematics that were not directly related to the function concept. In comparing the Israel and English students' responses, therefore, we are looking for evidence of improved answers to the survey questions across years, sustained levels of answer across years, and levels that appear to vary across years, and how these might relate to the curriculum.

Figure 1. Task 1: You are staying in a hotel on its 14^{th} floor. You are going to use the lift to go down to the parking level. The hotel has a ground level numbered zero, and there are several parking levels underneath the zero floor. The table below shows what floor you reach after a number of seconds.

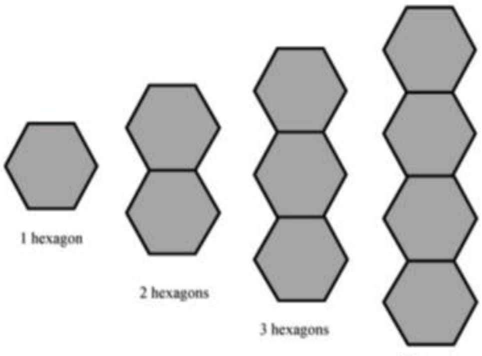
Number of seconds	Floor number
0	14
2	10
4	6
6	2
7	?

1.1 Where will the lift be after seven seconds? Explain your answer.

1.2 At what rate does the lift descend? Explain your answer.

Task 1 was designed to expose students' understanding of rate of change. Israeli teachers believed students would be able to answer these questions drawing on intuitive knowledge before they were formally taught in year 7, but UK teachers thought there would be difficulties as 'rate of change is not taught until year 12'. In fact, the responses in both countries showed a good understanding of rate of change, over 90% of students were correct for 1.2, even many who, in 1.1, assumed an arithmetic sequence. UK students made more errors of this latter kind (errors decreasing over age), possibly because the curriculum emphasises generalisation of sequences. However, overall success in all classes with this question was high, with sustained and improving success. We therefore claim that rate of change from data is a concept that is available to students whether formally taught or not.

Figure 2. Task 2 : For the following geometric pattern, there is a chain of regular hexagons:



2.1 For **1** hexagon the perimeter is **6**
For **3** hexagons the perimeter is **14**;
For **2** hexagons the perimeter is ______________
For **5** hexagons the perimeter is ______________

2.2 Describe the process for determining the perimeter for 100 hexagons, without knowing the perimeter for 99 hexagons.

2.3 Write a formula to describe the perimeter for any number of hexagons in the chain (it does not need to be simplified).

Students overwhelmingly used a correspondence approach, that is they tried to connect input and output variables, in task 2. This could be due to the design, because data were not given sequentially, nor in a data table, so slippage into recursive reasoning (perimeter goes up in 4s) was not readily, visually, available. Of the Israeli students choosing this approach, 41/44 presented a correct formula, mainly referring to the structure of the shapes to explain it, but the English proportion was 20/39. Almost two-thirds of the UK students made errors of reasoning that are widely reported in research: an incorrect proportional assumption (Van Dooren et al., 2009), e.g. 100 hexagons have a perimeter of 100 x 4, and their correspondence reasoning did not take into account either the construction of the structure, nor a correct deconstruction. *None* of the Israeli responses made this proportional error; their three incorrect responses were of inability to complete a deconstructive approach correctly (Rivera & Becker, 2008). The differences suggest that these errors arise from schooling rather than being inherent in the mathematics or in natural maturation.

We conjecture that Israeli students may have been more used to using an input-output correspondence approach successfully with linear functions, not limited to generalising sequences. An often-used approach to this is the use of 'function machines'. The inclusion of a function machine approach to constructing linear functions is advisory in Israel, this advice being found in a detailed curriculum document of over 26 pages (Israeli Ministry of Education, 2009) . This advice would not have affected our findings directly, but does indicate a commitment to a correspondence approach. In UK the long-standing related aim is to 'find the formula for the nth term in a sequence' so assessment materials tend to promulgate the sequential approach. Correspondence, in the form of function machines, appears in many UK textbooks but is not often attached to the task of generalising sequential data. Formal teaching in Israel about correspondence approaches appears to circumvent proportional assumptions, but does not 'cure' incomplete deconstruction. Given a task that is apparently about sequences, it does not seem to have occurred to many UK students to think in terms of function machines with which they may be familiar. UK students were more likely to be successful if they used a covariation approach (15/19). 8 Israeli students used this approach successfully out of the 11 who tried it.

The 'rate of change' approach to generating linear formulae is usually taught in the context of sequences, so step size of the independent variable does not have to be considered. Students who were successful with covariation were therefore either adapting a sequential approach, or adapting their understanding of rate of change. We conjecture that, whereas Israeli students were more likely to have been using a method they had learnt and worked with before, i.e. schooled knowledge, UK students were having to apply adaptive reasoning, this making the task considerably harder for them. Overall success in finding the formula was 92% in Israel compared to 50% in UK. A further difference between countries was that a few Israeli students used formal algebraic approaches that assume an underlying linear relationship: using two data pairs to find a linear function and using formulae associated with arithmetic sequences. Progression was evident in both countries in this task. We think that recognising and generalising linear functions is dependent on a range of taught approaches, and students who understand the purpose can use a *bricolage* of reasoning if they do not hit on a successful method initially.

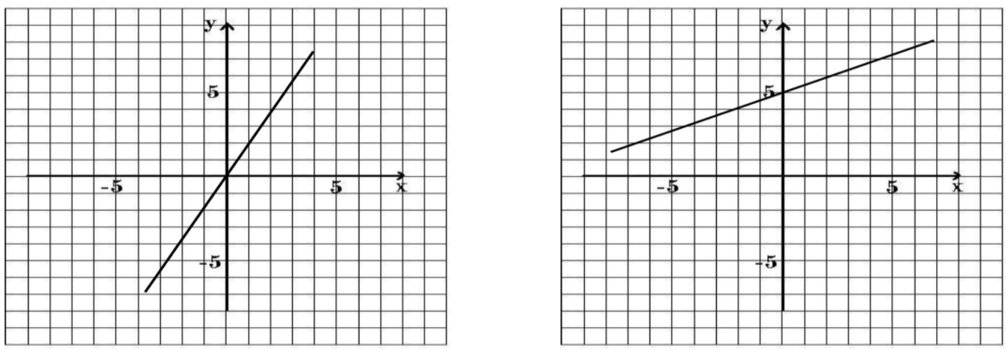
Figure 3. Task 3: Below are four straight lines. Two are in the form of equations, and two are in the form of graphs. Circle all those that are parallel to $y=2x+5$. There can be more than one answer.

1. $y = 2x + 9$

2. $y = 5x + 2$

3.

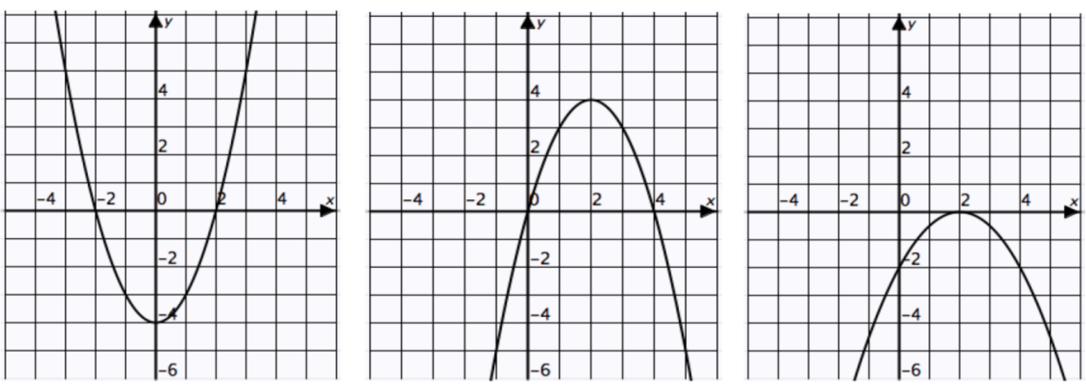
4.



Task 3 requires students to know algebraic and graphical representations of linear functions and compare their gradients, finding gradients using graphical representations. This is taught at roughly the same age in both countries. Informal knowledge, and even a geometrical understanding of 'parallel', are of little help because of the formality of the task presentation. Because these are ideas that are either remembered or forgotten we would not expect to see monotonic progression - rather one might expect a jump in success during the year in which these are taught, followed by sustained success or falling off. We see this in the Israeli data until the ideas are revisited in years 11 and 12. UK data is weaker in general but with clear year on year progress, with formal ideas appearing to first be used in year 8 then progressing until year 11. This suggests the ideas are revisited frequently in UK, but do not appear to become strong until year 13, in the specialist course.

Figure 4, Task 4: Three graphs appear to be quadratic. Students are asked to suggest ways in which A and B are similar and C is different; C and B are similar and A is different; etc.

A. B. C.



Task 4 (figure 4) was designed to expose what students thought the important features of quadratic graphs might be, thus supporting the necessary shift towards functions being objects in their own right, and graphs embodying their characteristics. It was also influenced by the research that claims students often treat graphs as pictures (Orton, Orton & Roper, 1999). By asking students to compare and classify features we made the task accessible to all students, whether or not they had studied quadratics, and also gave them the responsibility for finding characteristics. A key distinction that we found in the UK data was between students who described pictorial and visual features only and those who described the behaviour of the graphs, whether using formal or informal language. We found that some students moved towards using analytical features before they had been formally taught about quadratics. We conjectured that knowledge of graphs in general was informing the way they identified features of these new-to-them graphical objects. We therefore analysed both sets of data according to whether students used analytical features, and whether they did so using formal or informal descriptions. In Israel, there was more use of formal language and expressions for the chosen characteristics, possibly reflecting the more formal teaching (24% in the UK and 52% in Israel), but overall a similar proportion of students was identifying analytical features in both countries, 82% in UK and 77% in Israel, with Israeli students having more formal tools with which to describe them.

Choice of features was similar in both countries, with orientation and zeroes being the most frequent, followed by turning point. However, in UK formal treatment of these increased towards years 12 and 13, where students are specialising in mathematics and learning more formally about functions and calculus. In Israel formal descriptions of orientation and turning points increase in years 10 and 11 for similar reasons, but decrease significantly in year 12. Formal identification of zeroes described as intercepts (not roots) increases throughout. Therefore, accompanying an overall increasingly analytical approach to graphs, there is also the effect of when and how certain formal approaches are taught, and hence become available for students to use if not too much time has passed. We therefore claim that students can develop an understanding of the key features of graphs of functions through regular use in school, and formal teaching about these enables students to describe them in mathematical terms, but knowing WHAT to describe rather than HOW to describe it may develop more informally through wide experience since in all classes there is evidence of students doing so before formal teaching.

Task 5 asks students to match graphs and verbal descriptions of 4 situations, that is identifying suitable variables and how they are related, and choosing a graph that seemed to express this relation. It was described in Ayalon, Lerman & Watson (2013) and space does not permit reproducing it here. In both countries, success was situation specific, with some variables being easy to choose and other situations providing a choice of variables, or requiring students to grasp and use compound variables. One important feature in the design was that not all situations were amenable to having ‘time’ on the x-axis. To be successful, students needed to imagine the situations, identify variables and work out how they related - a combination of everyday and representational reasoning. All four situations depended on students having a sense of covariation, either in a naive sense of ‘going up and down’, or a more complex sense of varying the rate of change. Students in the two countries were remarkably similar in their approaches and difficulties in these tasks, and in the proportions of successful answers, 30% to 33% overall. Progress in UK was discernible as the proportion of students offering no analysis decreased, and the proportion being successful increased. In Israel there was no overall progression that we could discern, the strongest success rate being in year 10. Conceptual problems and approaches were similar in both countries, and older students were showing the same range of errors and difficulties as younger students in both countries. We conjecture from this that students’ capabilities with the relationship between verbal and graphical descriptions of phenomena are influenced more by features of the phenomena than by formal teaching or maturity. However, the UK curriculum which includes specific attention to applications appears to have a positive effect on students' progression, i.e. students can be schooled towards being more capable of making the connections.

Conclusion

We identified three kinds of 'non-progress': where responses are strong across all years, so an idea is retained (this applies to Israel class A tasks 1 and 2); where responses are erratic suggesting either differences in teaching or varied emphasis in different years, i.e. no maintained progress; where responses show variation across years - an idea has not developed given the curriculum and the teaching (such as Israel task 5).

The first two kinds are not too concerning so long as, in the second case, an idea can be revisited later and can then, through use and relevance, become more robust. The third kind is more concerning, as are some of the relative weaknesses between the countries. These higher performing Israeli students are less likely to make progress in realistic graph-matching tasks than their English counterparts, while the English students are less likely to enact the formal aspects of function understanding, due to having a less formal curriculum. All students displayed strengths in understanding rate of change, and in identifying key characteristics of graphs, whether formally taught or not. A worrying weakness in English students is that, with a strong curriculum focus on generalising spatial sequence data, they were less likely to be successful in constructing a formula when data was presented in a non-sequential form.

We are now hoping to learn more about when and how teachers use input-output or function machine approaches in both countries.

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