Exercises in mathematical imagining: setting out a teaching instrument that evokes imaginings and utilises visualisations in secondary school mathematics

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This paper sets out a teaching instrument which could be referred to as 'mental imagery exercises in mathematics' or 'exercises in mathematical visualisation'. Originally developed in the context of the teaching principles of the German-speaking countries, and then based on theories of the German mathematics education literature, the paper will reframe this task design and conclude that the term '*exercises in mathematical imagining*' is most appropriate. The fact that imagining, i.e. mentally forming and manipulating images, is personal means that this approach enables students to experience mathematics as commencing in their own minds. Teachers are also simultaneously able to gain insight into their students' thought processes through this approach. In contrast with mental arithmetic, this task design does not focus on training a particular ability. Rather it promotes a specific heuristic strategy, within which students first conceive of and imagine a mathematical topic, and then construct and explore visualisations in order to understand their implications. This paper should be read as a case study of the researcher's day-to-day teaching. It makes explicit a practice that has for many years produced positive results in terms of students developing mathematical understanding.

Keywords: mental imagery, visualisation, imagining, semiotics, task design, heuristic strategy, teaching instrument, secondary school mathematics

Preliminary remarks on terminology

Words can evoke images in our minds. Take the following task: *"Imagine a plastic cup rolling about on the floor. What is its path?"* (Mason, 2002, p.78) As the prompt "Imagine!" indicates, the intention is not that the problem should be addressed with concrete aids such as notes, sketches or other tools, but instead purely using imagination. I have developed many tasks of this type for my own mathematics teaching, and utilised them with students (in Years 10–13, i.e. ages 16–19) at a state secondary school in Switzerland. I subsequently investigated this practice and established it in theories on mathematics education developed and used in German-speaking countries, using the action research method (Weber, 2007; 2010).

In order to usefully address a topic, terms are needed that themselves refer to a certain, shared background (experiences, culture, history, theories, etc.). In the case of the 'Vorstellungsübung' of the rolling cup, the topic is not only mathematical, but also specifically relates to imagining and thinking. Concepts that convey such mental processes of course differ greatly across diverse linguistic and cultural areas. For this reason, new terminology is required that relates to an English-speaking context. This paper will first establish the necessary vocabulary in English for the 'Vorstellungsübungen' and re-ground this task design using the newly established terms. Secondly, it will indicate how tasks of this type can be used in secondary school teaching and thus become a teaching instrument.

The problem of 'Vorstellungen'

Terms such as 'Vorstellungen' and 'Anschauungen' have been much discussed in German philosophy—mostly connected with Kant's epistemology—and have come to be part of our every-day and scientific language. However, no equivalent concepts and terms seem to exist in English. Consequently, the famous title "Anschauliche Geometrie" (Hilbert & Cohn-Vossen, 1932) has been translated as "Geometry and the Imagination" (1952). Despite the term 'visualisable geometry' seeming to be more appropriate in terms of literal meaning (Zimmermann & Cunningham, 1991), 'imagination' was used instead. This word can also denote 'fantasy', a meaning that has nothing to do with 'anschaulich', and certainly does not reflect the intent of Hilbert and Cohn-Vossen; their original purpose was to *"give a presentation of geometry […] in its visual, intuitive aspects"* (1952, p.iii).

Conversely, English terms with regard to thinking and imagining present comparable problems when being translated into German. One example of this is Bruner's modes of representation for learning new concepts, i.e. his enactive, iconic and symbolic representation (Bruner, 1966). The English term 'representation' has both a mental (interior) and a physical (exterior) connotation. However, although 'representation' may be translated either as 'Vorstellung' or 'Darstellung', these German terms are mutually exclusive. In a sense, the German terms are opposing aspects—the interior and exterior—of the English term 'representation'. It is thus hardly surprising that many German authors writing on developmental psychology or education interpret Bruner's theory as being mainly focused on the physical aspect of representation, neglecting or even omitting the mental aspect of the three modes of representation.

Setting out 'Vorstellungsübungen' and 'Vorstellungen'

So how should we refer to tasks such as the rolling cup? A mental imagery exercise in mathematics? An exercise in visualisation? Or should we simply introduce the German term "mathematische Vorstellungsübung" (Weber, 2007; 2010) as a loanword?

'Mental' conveys that a given problem should be solved without the use of concrete objects and actions, but this adjective also triggers the idea of mental arithmetic when used in the context of school-level mathematics. However, this is problematic. While mental arithmetic exercises relate to training in order to develop the ability to reproduce computational results quickly, this ability is not the focus of the tasks discussed here. There is also the fact that the term 'imagery' is associated with static 'mind-images'; the task design discussed here, however, should ideally also involve dynamism, which is to say in the form of 'mind-actions'. There is also a debate in cognitive psychology regarding imagery. This debate concerns whether thought is imagistic, i.e. whether or not it is based on mental images, and whether such images equate to real-world ones. It is therefore clear that terminology such as 'mental imagery exercise' is ultimately unsuitable if associations like these are to be avoided.

The term 'exercise in mathematical visualisation' also has its pitfalls. 'Visualisation' signifies forming images in one's mind. But that is not all; it also specifically signifies forming images on a piece of paper or on an electronic screen. This ambiguity makes misunderstandings inevitable from the outset. Then, because 'visualisation' is also used in the context of the intelligence construct ('visualisation ability'), 'exercise in visualisation' could again be misleading in that it may indicate training. As a final point, the term's connotation is particularly focused on the visual, although tasks of this type are not limited to visual modality, i.e. it is possible to perceive in a tactile way how the rolling cup's conical shape means it is set in motion and urged out of a straight path. This means that 'exercise in visualisation' is as problematic as 'mental imagery exercise'. However, the term 'visualisation' has a certain meaning in our context, as indicated by the title of this paper, and will be discussed in the following section. (The fact that the term 'visualisation' and related terms such as 'imagery' are open to a wide variety of interpretations has been shown recently in a literary research project conducted by Phillips, Norris, and Macnab (2010).)

As such, 'exercises in mathematical imagining' seems to be more pertinent whilst also being more natural. First of all, 'imagining' seizes upon the instruction "Imagine!" and therefore refers to imagination. 'Imagining' is not only more colloquial, but also has a broader meaning than 'visualisation' and 'mental imagery'; for tasks of this type, it is less a case of calling images to mind that are as close as possible to actual visual perception ('mental imagery'), and more the creative act of forming and manipulating imaginings (Colombo, 2012). Ultimately, imagining and imagination are always associated with fantasy and intuition (see above), and this includes the individual nature of these types of processes. Further, this corresponds with the intention that had previously informed and guided my teaching; my intention was to enable the students to access mathematics on a personal level using visual imagining, even if this access was diffuse and vague. For these reasons, this paper uses the term '*exercises in mathematical imagining*' (as already did Conway, Doyle, Gilman, & Thurston, 1994, p.28).

As the following sections will show, tasks such as the rolling cup do not only encourage students to imagine and thus create *imaginings* in the form of mind-images and mind-actions. They also promote the use of an individual's own imaginings as a basis for reasoning in the classroom. From a didactic perspective, tasks of this type can therefore encourage students to use imagining and imaginings for heuristic and therefore epistemic purposes, and to prove and understand mathematical facts. Although exercises in imagining could be seen to develop mathematical competence in some way, their scope of impact is much more direct.

Visualisations in mathematics and mathematics education

Images that visualise something are of interest not only in mathematics education but also in mathematics itself. In ancient geometry, sketches and drawings were integral to mathematics. Today, many mathematicians consider visualisations in the form of figures, schematic diagrams, graphs, etc. to be useful as heuristic tools, i.e. as motivation for and evidence of mathematical statements. Some mathematicians even use them as a substantial part of the result ('proof without words'), i.e. as a justification and proof of mathematical statements (Hanna & Sidoli, 2007; Phillips et al., 2010).

Mathematics educators are interested in how to teach approaches for finding mathematical results. Every-day practice in teaching and problem-solving shows that visualising can sometimes be helpful (Posamentier & Schulz, 1996). At the same time, it is well-known that visualisations can mislead or even hinder an answer being found. Not only this, but visualisations also seem to work differently for different students (Presmeg, 1986). Accordingly, a review of forty empirical studies focusing on visualisations in mathematics classrooms shows that mathematics education is not at all certain of the value of visualisations: "*The research indeed is ambiguous and often contradictory.*" (Phillips et al., 2010, p.50). Nevertheless, it is worth considering visualisations from a slightly different, namely semiotic, perspective in order to offer a reinterpretation of exercises in mathematical imagining.

Framing visualisations in semiotics

Alongside standardised visualisations, mathematics also uses other representations such as digits, symbols, etc. Following Duval, such mathematical representations must never be confused with a mathematical object. Although the contrast between mental and physical representations may be important for the learning of physics or biology, this is not the case for the learning of mathematics. If it is agreed that mathematical objects can never be physically and directly perceived, the contrast between *objects* and *representations* is much more relevant: mathematical representations of any kind are purely semiotic signs and, as such, vehicles to convey information regarding the objects (Duval, 2006).

A semiotic framework of this type explains why visualisations are never self-explanatory. They instead always require an individual to perform cognitive activity that is neither mental nor physical, but *semiotic*: it is necessary to know to what mathematical object a specific visualisation refers to; what meaning the visualisation has. Further, as Duval states, although school mathematics trains transforming representations (e.g. $2x + 3 = 4 \rightarrow x = ½$), understanding mathematics is not possible without being able to, using Duval's terminology, "convert" such representations. Only through converting a specific representation (e.g. $y = x^2$) into other, equivalent representations (for example a graph or a table of values) is it possible to learn which mathematical object these different representations refer to (the parabola). Meaning-making is possible on the basis of transforming and converting representations.

In his analysis, Duval refers only to scientific textbooks and journals and therefore only to standardised visualisations and representations. In the course of an exercise in imagining, however, personal imaginings and visualisations are constructed, which are the result of the student's individual work on a mathematical question. It is useful to look at Presmeg's work here. In an empirical study, she investigated what types of images students produced in order to solve problems. As a result, she constructed five classes of images and investigated their efficacy (Presmeg, 1986). In her later work, she also took into account externally presented visualisations (referred to as "inscriptions"), and again recorded them as she had done with visual images, namely semiotically as "sign vehicles" (2006). As a result, the focus of interest shifts towards the visualisations students produce and the references in the sense of meanings they ascribe to their visualisations, and therefore how they make use of them to answer a mathematical question.

Setting out how to evoke imaginings and utilise visualisations

Due to my scientific socialisation, my Ph.D. studies on exercises in imagining followed the educational theories developed in the German-speaking countries during the 1990s. Accordingly, I concentrated on the imagining-based side of visualisations (Weber, 2007). The didactic value of this teaching instrument could then be located in, among other areas, the formation of conceptual models, i.e. in creating robust imaginings referred to as "Grundvorstellungen" (Kleine, Jordan, & Harvey, 2005).

The semiotic view on visualisations as signs, however, allows a different, more pragmatic view on exercises in imagining. Students are no longer required to have clear imaginings, but produce and work with written representations of their own imaginings, i.e. *visualisations*. The crucial point now is not if a student imagines a described fact exactly as described but if he or she creates visualisations that are or can be made productive in the sense of answering a mathematical question.

Utilising students' visualisations: a classroom episode

Irrespective of rudimentary definitions, theoretical debates and many open questions, it is possible to deal with exercises in imagining in the classroom. Literature relating to mathematics teaching refers in different ways to the value of imagining (see e.g. Mason, 2002) or written visualisations (see e.g. Arcavi, 2003). However, it makes no statement of any kind as to how teaching can utilise imaginings and visualisations in order to understand mathematics; how must imaginings and visualisations be manipulated—i.e. "transformed" and "converted"—in such a way that they serve as a workable basis for developing a mathematical argument or answer? And how can students be guided towards this? Presmeg (2006) refers to this issue as follows: "*How can teachers help learners to make connections between idiosyncratic visual imagery and inscriptions, and conventional mathematical processes and notations?"* (p.25)

In conclusion and to provide insight into this process, a classroom episode is presented. It starts with the following words (Weber, 2010, pp.136–137):

> Please close your eyes. … Imagine a ladder in a light and empty room. Take the ladder and lean it closely against the wall. Imagine yourself at the left-hand side of the ladder and lean your left shoulder against the wall. You now see only one side rail of the ladder leaning against the wall of the room. A lamp is attached to the middle of the side rail facing you. Darken the room and turn the lamp on. You see it shining as a point of light. The bottom end of the ladder begins to slide slowly along the floor to the right, away from the wall. The top end of the ladder continues to touch the wall and slides down it. When it touches the floor, it stops sliding and comes to rest. What is the shape of the trace of light drawn by the lamp as a result of the ladder sliding? What did you imagine during the exercise?

Following a mathematical imagining such as this, most of the students felt very much involved with their own, individual ideas and wished to talk about their imaginings, conjectures and difficulties. If this strong impulse is used and thoughts are put into writing, a sequence of lessons can be designed around it. First, the imaginings were established in writing in students' journals so that they did not dissipate and became visualisations. To this end, I posed the following questions: a) *What is the shape of the trace of light 'drawn' by the lamp as the ladder moves? Describe your conjectures and draw a sketch*. b) *Note down all of the mind-images and mind-actions you imagined during the exercise.* The next question was intended to encourage students to reflect on and assess their visualisations: c) *Which of your mind-images and mind-actions were useful with regard to your conjectures? Which ones hindered you?*

The shape of the path of the lamp was generally assumed to be convex, and sometimes concave or straight. Images that were too concrete or had characteristics of a real-world picture (such as a soft carpet on the floor), were found to be hindering. On the other hand, some students considered it useful to consciously manipulate their imaginings and set up the problem again. One student even wrote:

> I had actually imagined [a sketch of a convex curve] because to begin with, I was actually sure that it would have to be shaped somehow something like this. But I hadn't thought of the shape of the falling ladder. Falling and sliding is the same.

This means that the student had been successful in manipulating his original, hindering visualisation. Slightly later on, he integrated the sliding and the falling ladder into the image of a pair of scissors, meaning that he had found a plausible argument for the circular shape of the light path. On the basis of this conceptual image, he was then also able to prove the elliptical shapes of the general light paths.

Going back to the teaching process, once the students had formulated and assessed their assumptions in their journals, they swap journals with one another. A change of perspective was introduced: *Read what your classmate has written in his or her journal and give feedback:* d) *Which visualisation do you think is of particular interest?* e) *Which visualisations are useful for answering question a)? In what way?*

After having had their journal returned to them, the following question was finally posed: f) *How do you now answer question a)? What are your reasons for this?* In the case of the sliding ladder, this question can stimulate the analysis of the dominance of the convex curve. One student wrote: *"The image of the hyperbola came from concentrating on the ladder and the point: the limiting curve of the ladder's images is a hyperbola."* In other words the supposedly convex form of the light curve results from the envelope of all ladder positions. Understanding the origin of a misleading visualisation can be the first step to reducing its dominance.

All of this happened in a single lesson. It finished with an experiment: one student moved a large ruler against the board in a movement similar to the sliding ladder, while another student guided a piece of chalk in line with the centre of the ruler. The sketch on the board rendered it clear that the searched-for curve is not convex, but could in fact be circular. (More information on the overall knowledge-creation process, in which students developed mathematical proofs on the basis of their visualisations, is given in Weber (2010), illustrated with students' visualisations.)

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