Learning mathematical model-making

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The processes for developing, sharing, and using models - idealized, partial, purposeful descriptions - in the workplace and everyday life are reviewed. They seem to be intuitive. But we are reluctant to replace our out-of-date models, leaving this to the next generation. We live in an era of unprecedented rate of change with corresponding need for model-making on an increasing scale and so the possibility of enhancing, in secondary education, the intuitive modeling skills of the next generation should be investigated. As a first step, a controlled experiment in learning mathematical model-making at secondary level, by means of one-day investigative workshops is proposed.

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Viewpoints, models and modeling

Models are purposeful *descriptions.* They aim to capture and then impart *insight* – i.e. understanding - regarding something in the real-world in which we live, for example a physical object, a process, or a relationship. The description provided by a model is *abstract* in that it is *idealised and partial*, only containing details relevant to *the purpose of the model.* Thus it is a tool for managing complexity. Models start out as mental constructs or proto-models. For example, various people looking at an everyday object such as a tree - a landscape artist, a tree surgeon, a timber merchant, and a small boy intent on climbing the tree - all see the same physical tree but each also has his proto-model of that tree from his particular *viewpoint*. The individuals share an interest in the tree - they form a *community of interest* - but their viewpoints, and hence proto-models, are different because they regard the tree with different *purposes* in mind; they have different *requirements* of it. We might expect individuals with the same, or at least similar, viewpoints to have similar proto-models - such a collection of "like-minded" individuals may be called a *community of practice.* We should be alert to diversities of viewpoint and aware that a community of interest does not equate to a community of practice. If we ask for a description - in speech, or drawing, or text, or whatever - of a proto-model, we get an actual model which can then be compared, critiqued, improved, and shared within a community of practice. This process, beginning with (somehow) acquiring a proto-model through to its adoption, is called *modeling*.

Acquiring our models

During our lifetime each of us surely acquires a personal repertory of proto-models encapsulating our understanding of the world. We learn some by example from peers and parents, and construct some through our own observation and experience; a new situation to be negotiated requires an addition or a modification to our repertory. But since there is relatively little formal teaching of them we may deduce that the learning is generally *intuitive*. Various media are used for communicating and recording models including natural language, diagrams, algorithms and mathematics. Published models may have *users* who had no part in its making. We now review modeling and model-use in three domains: the Workplace, Everyday-life and Education.

Workplaces, if they are large organisations, house several occupations collectively forming a community of interest. Each occupation is a community of practice with its knowledge and practices encapsulated in its own set of models recorded and expressed in characteristic form. Modeling is generally a minority activity, the majority of workers being model-users only, without participating in the making or adoption processes. Often the form of the models is mathematical. Scientific knowledge, for example, is encapsulated in a set of inter-related models, called *theories*, whereas the sets of models owned by most professions - accountancy architecture, engineering, and medicine - emphasise practice, and for workers in high productivity industries they are all about practices. Recognition that the multiple viewpoints/requirements within a community of interest give rise to multiple models can be important and failure to do so underlay many large information technology (IT) project disasters. Software engineering from the 1980s onwards, for example (Mullery, 1985), developed techniques for capturing this diversity.

In everyday-life we are all users of models, and professional modelers are a minority. The media bombard us with a range of models concerning belief and ethics, lifestyle, relationships and more, all clamoring for adoption. This contrasts with our relatively sheltered workplace lives where we share a limited range of models with colleagues. Few everyday models are mathematical; the exceptions relate to money and time management and in some cases leisure, most obviously DIY. There is some scope and purpose in these areas for making personal models.

School provides an introduction to scientific models as well as some training in the various media used for communicating models. And while preparation for life in our world/society/civilisation is often the official goal of general education, it has not generally been expressed in terms of acquiring models. Training for a career in any profession involves immersion in that profession's set of models and, while the term is not always used, its adoption is becoming more common, as evidenced for example by increasing reference in financial reporting to "business models".

In an unchanging environment, after reaching maturity we might have no need to acquire further concepts or models or to modify those we are accustomed to using. But social and technological changes render them obsolescent and they will need updating; continuing change will require them to be re-made; new circumstances will require new ones to be made and learned. Our children are noticeably more comfortable with new models - the advantage of a *tabula rasa*. But in our era of unprecedented technology-induced change, the "churn" in models is greater than ever before with new models emerging, more frequent updating, etc, and the consequence that users must also make frequent, and uncomfortable, model changes.

The author's workplace trajectory - physics, measurement and instrumentation, electronics product innovation, computer science and software engineering - during the decades of the silicon and then software digital technology revolutions, compelled participation in the construction, refinement, and early use of a succession of new models. Despite or perhaps because of an absence of precedents, it was a confusing but sometimes satisfying and even exhilarating experience, not least because of *insights* - minor eureka moments. These occurred when several related instances prompted a generalization – a formulation of a new model. Until now, education has not generally prepared us for modeling and those, like the writer, who became professional modelers, proceeded intuitively which felt like "muddling through". But the pace of information technology-driven innovation and the associated need for modeling skills is now greater than ever, suggesting roles for education at all levels.

Mathematical models (MMs)

MMs are particularly *powerful* models because they can be written down in mathematical language and hence communicated, but also because they can be *structured* (mathematics is a language of patterns) and so can accommodate detail without loss of clarity and are amenable to deduction so that, for example, *instances* of a general model may be generated by substitution of particular parameter values.

A MM describes a property of something in the real-world in terms of a topic in pure mathematics. Some examples are:

- 1/72 scale toy model of an aeroplane (ratio)
- A savings account (compound interest: interest rate parameter distinguishes accounts)
- A hollow rectangular box model of a room (geometry of rectilinear shapes: length, width, and height parameters generate rooms of various sizes)
- Electrostatic and gravitational forces (geometry of the sphere)
- Mendel's genetic theory (discrete probability distribution)
- Radioactivity (continuous probability distribution: parameter values give the distribution instances- lifetimes- for different isotopes)
- Physics knowledge generally (encapsulated in a collection of related MMs, principally differential equations)
- Chomsky's syntax analysis (recursion)

Sometimes MMs can be made intuitively, without formal mathematics knowledge, because they are visual and our brains recognise and process patterns. Some well-known examples are:

- London's Tube map (set theory)
- Mendeleev's Periodic table of the elements (two variables determine horizontal and vertical locations in a rectangular array)
- Family trees (hierarchy).

Using a workplace MM typically requires particular mathematical and professional knowledge; for example a structural engineer choosing a steel beam to support a wall knows how to consult a model in the form of tables of loads and dimensions, whereas a Tube traveler can plan his journey without formal mathematics. Incidentally, the Tube map - now a London icon - provides an example of the importance of viewpoints in relation to models; the inventor's contract was terminated because his manager was unhappy that the map did not respect scale - two Wimbledon stations on different lines are shown far apart!

Learning MM-making

The mathematics education community is currently interested in MM-making and its processes and its relation to traditional applied mathematics problem solving. Blum et al. (2007) have drawn a distinction between the two which I paraphrase in terms of the different viewpoints of a mathematician and an inhabitant of a real-world domain when confronting a domain problem: (a) the mathematician selects an appropriate mathematical topic and *applies* it to the problem; (b) the real-world inhabitant *models* the problem using a topic from his mathematics knowledge.

MM-making has long been taught in some HE mathematics courses (Bender, 1978) and typically follow (a). Traditional school mathematics gives learners the viewpoint of mathematicians so that, after studying a fresh topic in pure mathematics, they apply it in the solution of word-problems in a (imagined) real-world. In traditional applied mathematics in schools, learners encounter a few standard examples of generic MMs; equations of motion, statistical analysis of data; and then instantiate them in real-world exercises. But this is *use* of MMs, not MM-making. A recent textbook (Maasz & O'Donoghue, 2011) reflecting current interest in MM-making at secondary level adopts an investigative approach and is closer to (b). In school science the historical processes whereby significant MMs (Newton's theory of gravitation, Mendel's pea breeding experiments) were developed get some attention, which is evidently akin to process (b).

The processes have complementary weaknesses and strengths. The real-world inhabitant has a perspective view of the problem; it arises in his domain, he owns it and he can distinguish its important from its peripheral aspects, but he may not have the particular mathematics for modeling it. The mathematician has the mathematics but, unless he also inhabits the problem domain, lacks perspective and cannot make a start without direction. A famous example from a workplace domain (science) illustrates the problem and a solution through collaboration: when Einstein was struggling to formulate his General Theory of Relativity, he had an intuitive grasp of the physics but not enough non-Euclidean geometry to make his MM and he needed help from a pure mathematician. There is a further significant difference between the two processes. The mathematician's viewpoint on modeling is essentially deductive - once he has a MM describing a general problem situation he can deduce particular cases, whereas non-mathematicians are likely to proceed inductively- first making a MM for a particular instance of the problem, then perhaps generalising it.

There are some good reasons for getting secondary level learners to do some MM-making. They may enjoy it more than traditional mathematics, they may find it insightful, it may motivate further mathematics learning, and some may find they are good at it and want to do more. And in our era of continuing rapid socio-technological changes the requirement for continual MM-making and remaking should ensure good jobs for talented modelers. Mathematicians at the secondary school level know some mathematics and also know about their everyday-world. Evidently we should seek problems in this domain for learners at secondary level to develop their MM-making.

MM-making project

A secondary level MM-making project (Moped) is proposed, with the participating learners possessing both the needed mathematical and domain knowledge. The project concerns money-management and has the form of a 1-day workshop, with both learning and research goals. Goals for the learners are: (1) an intense and enjoyable learning experience in which some of their (perhaps latent) mathematical knowledge is drawn-out and used to model a meaningful everyday-world financial dilemma and its resolution, which is then generalized; (2) appreciation that they own some useful mathematics knowledge and MM-making is interesting and can be mentally rewarding.

Moped: a Money-Management Mathematical Model-making Workshop

Scenario: School-leaver in first job, travelling to and from work by bus. Journey-time is uncertain (involves two buses with unreliable connection); Journey-cost is high. For both reasons considering buying moped/scooter instead of travelling by bus.
In this workshop we will focus only on the costs for the two modes of travel, make a model comparing them, and then see where else the model might take us.

Session 1 Model-Making

Data collection: Get cost data - both modes - from local knowledge and Internet research
Note patterns of payments:
- [Small and frequent (both modes: Bus-pass/petrol)]
- [Large- one-off or occasional (Moped only)]

Idealise the patterns: [Assume *regular* payments]
Choose units: [Time in months, Payments in £]
Make spreadsheet and enter data

Session 2 Relating expenditure patterns to mathematics knowledge

Use spreadsheet to generate cumulative expenditure patterns
Describe these patterns:
- Bus mode: [grows *linearly* from zero]
- Moped mode: [grows *linearly* between annual steps]

Match patterns with corresponding pure mathematics topic: [linear relations]
Map variables and parameters [in linear relations] *onto those in the scenario:*
- [display symbolic graphical representations]

Session 3 Exploring the model

Time-to-break-even? [When expenditure on Bus and Moped are equal]
What-ifs: *Use spreadsheet to vary parameter values*:
- *What does this do to Time-to-break-even?*
- *Can there be more than one Time-to-break-even?*

Session 4 Reflecting on the making, the model, and the mathematics

Cumulative expenditure:
- Why is this comparator better than incremental expenditure?

Time-to-break-even emerged as a simple cost comparator for the two modes:
- Was this clear before modeling?
- Why is this shown by the intersection of the two graphs?

Variables and parameters: What is the difference?
Cost-of-ownership - How complete is our analysis? What about:
- *Opportunity cost?* [other uses for Moped expenditure monies,
  e.g. Moped expenditure into savings account]
- *Other benefits?*
  [Moped use for other journeys besides travel to work]
- *Risk?* [Moped stolen or damaged]

Table 1. Structure of the proposed 1-day Project Workshop showing *Tasks* and [Hints]

It is envisaged the workshop will take place at the end of the summer term after public examinations (GCSE, AS-level, or A-level). The proposed format is derived from the author's experience of three intense practical learning situations in HE: 6-hour science practical examination, group projects in IT courses, IT-mediated collaborative design. The format is as follows. The class is split into groups of four, each group sitting at a table with a microphone for recording group conversation and a PC equipped for Internet search (e.g. Google), a spreadsheet, and a task list. The workshop structure is shown in Table 1. The MM-making process is summarized as follows. The first session is given over to data research and collection- and inputting data into to the spreadsheet as pairs of data points. In the second session the two data patterns are modeled by linear relations, albeit somewhat unfamiliar in that one pattern is linear but with discontinuous changes at regular intervals. In the third session varying parameter values generate a set of instances of related data patterns and a generic comparator - "break-even" - is identified. The final session is given over to reflection and potential further generalisation to cover broader "cost-of-ownership" issues.

The chosen financial dilemma is intended to be realistic and one all the learners will identify with, giving them a common viewpoint, so that each group should function as a community of practice collaborating in the construction of a model. Learners should have equal opportunity to contribute and so no *role specialization* - no group-leader, secretary, researcher roles (in contrast with most group projects). The teacher/supervisor's role is to ensure steady progress through the task list. The project is intended to be investigative. While interventions will, hopefully, be unnecessary, the teacher has a dictaphone for recording any prompts. The progress of each group is monitored and recorded: their conversation on audio tape, their mathematical working on their spreadsheet, their written responses to prompts in a Word document. Hopefully, the recorded data will show. for the class as a whole, by group, and by individual learner, how well the workshop delivered the intended significant mathematical and modeling experience.

Mathematical modeling, as distinct from applied mathematics, is not yet taught in our secondary schools, so it would be timely to monitor how close to being modelers their present mathematical experience takes these learners. The workshop is portable - reproducible in form and structure - with other classes in other schools, at other levels, so that in combination, statistically significant results should be obtainable. Thus it offers a reliable reference-point measure of MM-making abilities after secondary mathematics, against which the effects of further intervention may be observed.

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