# Lecturer's use of genericity across examples in mathematics tutoring

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In this paper, I report early findings on characteristics of university mathematics teaching in small group tutorials, and in particular, I focus on a *lecturer's use of generic examples* in tutoring. The characterization draws on data from single tutorials of 26 lecturers and on data from a systematic study of tutorials of 3 of the 26 lecturers for more than one semester. A teaching episode has been selected from this data as a paradigmatic case to illuminate the lecturer's use of genericity across examples, supported by data from interviews in which her underlying considerations emerge.

**Keywords: genericity across examples, generic examples, small group tutorials, university mathematics teaching, injectivity**

## Introduction

Although there are many studies focusing on understanding and improving mathematics teaching and learning at school level, there are relatively few such studies at university level and particularly in the small group tutorial setting in which this study is situated. A systematic literature review from Speer, Smith and Horvath (2010) categorised published scholarship in university mathematics teaching and showed no systematic data collection and analysis focusing on teachers and teaching. However, research on university teaching practice can produce resources that novice and experienced university teachers might access for professional development, and it can improve the design of teacher development programs and materials (Speer et al., 2010). This paper contributes by focusing on mathematics teaching at university level and offering new insights into small group tutorial teaching in mathematics. It focuses on a potential characteristic of such teaching, a *lecturer's use of generic examples* and on how this use is linked with the sources of knowledge discerned in her teaching.

A small number of studies into university mathematics teaching in the context of small group tutorials has explored teaching in relation to students' learning outcomes and difficulties. Jaworski (2003) distinguished tutors' exposition patterns as the main teaching aspect, with the most prevalent ones being *tutor explanation, tutor as expert* and forms of *tutor questioning.* Nardi, Jaworski and Hegedus (2005) produced a spectrum of tutor's pedagogical awareness (SPA) with four dimensions: *Naive and Dismissive*, *Intuitive and Questioning*, *Reflective and Analytic* and *Confident and Articulate*. In the production of the SPA, Nardi et al. (2005, p. 302) discerned tutor's strategies for students' overcoming of learning difficulties, and their use of "*generic examples to create and enrich concept images of newly introduced concepts*" were among these strategies. In particular, they decompose the role of generic examples as the ones that are "epistemologically rich"; they incorporate the essential features of concepts and clarify pitfalls. Nardi (2008) investigated mathematicians' perceptions of their tutees' learning and reflections on their teaching practices. Furthermore, in the context of lectures, Petropoulou, Potari and Zachariades

(2011) manifest lecturers' strategies to construct mathematical meaning in teaching, and among them, a strategy which resembles the use of generic examples: lecturer's use of examples to illustrate critical characteristics of concepts.

## Literature review on generic examples

Mason and Pimm (1984) introduce the notion of generic example as the specialization/example that refers to a class of objects. In this sense, *kleenex* stands for tissue and *hoover* for vacuum cleaner. The key features of tissues or vacuum cleaners are emphasised in the example and so, the specialisations are presented initially as members of the class and eventually come to represent the whole class. Furthermore, with regard to elementary algebra, Mason and Pimm argue that the context in which the example is set recommends its specificity, particularity, genericity or generality. So, a specific example of even numbers is the number 6; however, it can also be seen as a generic example of even numbers, if the key feature of evenness is stressed (e.g. by rewriting it as 2×3) and the irrelevant features (e.g. divisible by 3) are ignored. Another generic example of even numbers is $2N$, which carries more layers of generality within it than 6, since it stresses the key features that make it generic of even numbers and ignores the irrelevant features regarding $N$. Depending on the use of language (e.g. definite or indefinite articles), $2N$ can be seen as general or particular example of even numbers as well. To this end, *the* even number $2N$ can be perceived as a particular but not specific example of even numbers and *any* even number $2N$ can be perceived as a general example of even numbers. The definition of a generic example, with which Mason and Pimm (1984, p.287) conclude, is "a generic example is an actual example, but one presented in such a way [by stressing and ignoring key features] as to bring out its intended role as the carrier of the general." In other words, the generic example is an example that is presented so that its key features bring to mind the large class. However, Mason and Pimm stress that although lecturers see the genericity of the example they use, they are often not clear about it in teaching, and it can be that students just focus on the particular example ignoring the large class.

A few years after Mason and Pimm's contribution to generic examples, Lakoff (1987) focuses on the production and the internal structure of the large class. Rules or a general principal which apply in a particular member of the class take this particular member as input and yields the entire class as output, thereby producing and characterizing the class. This particular member is the prototypical example of the class. For instance, considering a sparrow as a particular member that characterizes the class *bird*; the rules that apply in the class *bird* can be the possession of wings as well as the ability to fly, or alternatively the general principle of the class *bird* is that of similarity with a sparrow. However, the aforementioned rules or general principle do not indicate that a penguin is a member of the class *bird* since a penguin uses its wings to swim. A common pitfall regarding the use of prototypical examples is that if another example (member of the same class as the prototypical example) does not comply with the prototypical example, then the new example cannot be recognised as member of the same class. Lakoff (1987, p.43) distinguishes the "goodness-of-example ratings" within a class, rating the prototypical examples as 'good' examples of the class in terms of reflecting the internal structure of the class. For instance, a sparrow is a 'better' example of the class *bird* than a penguin is; for instance, most birds as well as a sparrow can fly. So, Lakoff (1987) characterises the prototypical examples as being 'central' in the internal structure of the class, meaning that the prototypical examples are 'good' examples of the class. In the context of geometry, I think of a square as a prototypical example of the class of quadrilaterals because of its very special properties (e.g. all sides have the same length, all angles are right). So, if my 'good' example of the class of quadrilaterals is a square, I may not conceive that a rhombus is a quadrilateral since although it has four sides of equal length, its angles are not necessarily equal. I may also not regard a kite as a quadrilateral, since it has neither of the two aforementioned special properties of a square. The difference between prototypical examples and Mason and Pimm's (1984) generic examples is that according to Lakoff (1987), the prototypical examples are 'superficial', or put another way, they do not carry layers of generality within them. Presmeg (1985, cited in Presmeg 1992, p. 600) comments that students use examples to recall theorems in Euclidean geometry; however, when the former examples fail to be prototypical ones or when students focus on the irrelevant features of the examples, which they should have ignored, learning difficulties arise.

In conversation after the BSRLM session, Rowland drew our attention to dimensions of variation (Watson & Mason, 2005, adapted from Marton & Booth, 1997) as a safeguard against overgeneralising from prototypical examples. According to Watson and Mason (2005), examples are "anything from which a learner might generalise" (p.3). Furthermore, "examples learners produce arise from a small pool of ideas that simply appear in response to particular tasks in particular situations" (2005, p.ix). They call these pools example spaces and their structure consists of dimensions of variation. For Marton and Booth (1997):

> To experience a particular situation in terms of general aspects, we have to experience the general aspects. These aspects correspond to *dimensions of variation*. That which we observe in a specific situation we tacitly experience as values in those dimensions. (p.108)

Watson and Mason (2005) stress that the variation constitutes a generality, which can be seen through examples lying in these different dimensions. The learner might be aware of these examples/values or s/he can regard them as extensions of the range of permissible change of his/her original dimensions of variation in his/her example space (Watson & Mason, 2005). However, prototypical examples might eliminate this range of permissible change or in other words, might not permit the integration of other examples/values in this dimension.

Mason and Pimm (1984) and Balacheff (1988) stress the importance of the presentation of the generic character of a generic example so that the mathematically immature student can conceive it or the sceptical partner/recipient does not doubt for it. Rowland (2002) explains further:

> A *transparent presentation* of the [generic] example is such that *analogy* with other instances is readily achieved, and their truth is thereby made manifest. Ultimately the audience can conceive of *no possible instance* in which the analogy could not be achieved (p. 161).

Rowland (2002) offers a range of generic examples in number theory both from literature and his own university teaching. His considerations concern pedagogical intentions of the use of generic examples in classroom and textbooks: "to convince" (2002, p. 157) and "to engage" students (2002, p. 176). Rowland (1999, p. 25) defines the generic example as "a confirming instance of a proposition, carefully presented so as to provide insight as to *why* the proposition holds true for that single instance", thereby explaining what the generic example should mean to students so that they can see the genericity in generic examples and not just the particular example. Concerning the issue of how to link generic understanding and general exposition, Rowland (2002) concludes giving three suggestions to teachers, whereby the second is to invite students to connect the generic argument with another particular case so that they can see the generic character of the generic example.

## Methodology

This study is part of an ongoing PhD project being conducted in small group tutorials at an English university. Small group tutorials are 50 minute weekly sessions consisting of 5 to 8 first year undergraduate students as well as a lecturer in modules offered by the mathematics department. The modules usually tutored are analysis and linear algebra. Zenobia, a research mathematician, is one of the 26 lecturer-participants and does not prepare a design for her tutorials. Data, consisting of observation notes and transcriptions of both her audio-recorded small group tutorials and follow up interviews, were collected from her tutorials for more than one semester. The interviews are discussions with her about her thinking behind the teaching actions in these tutorials. The characteristics emerge as codes through a grounded analytical approach and the unit of analysis is the identification of teaching episodes. A transcribed teaching episode is presented and analysed below, being selected from a vast amount of data as a paradigmatic case, or in other words as a form of generic example, of the characteristic *lecturer's use of generic examples*.

## Analysis of a teaching episode concerning injectivity

The small group tutorial, of which the following teaching episode is part, was about calculus revision towards the semester exams which were approaching. While Zenobia and the students were working on a past paper and before they worked on *injectivity* in an exam question, the following discussion took place:

Zen: Are there any kinds of functions that you know are going to be injective,
for instance? Is there anything about a function that you… Ok. So, let's
draw some functions on the board, shall we? So, here's an example of a
function. [The graph of $f(x)=x^2$.] And here's another example of a
function. [The graph of f(x)=sin(x).] And here's an example of a function
[the graph of f(x)=ln(x)], and here's an example of a function [the graph
f(x)=x]. So, if you wanted to determine some domains on which all of
these are injective, how would you do it? How would you do it for this
one? [Zen. points to the graph of $f(x)=x^2$].
S1: From 0 to ∞. [Zen. draws a red line from 0 to ∞ to show the domain on
which $f(x)=x^2$ is injective.]
Zen: Right. This is definitely not injective on the whole thing, right? Because
if I go off in opposite directions, I'm going to the same thing. Ok. But if I
go from here on, that's injective, right? Ok.

In this episode, Zenobia is starting off by asking the students whether they know any *kinds* of functions that are going to be injective [lines 1-2]. Her request for *kinds* of functions is significant, since she asks for examples that are representatives of classes of functions. She is then looking at the students and waiting; however, since the students' feedback is null, she is devising the examples herself: the parabola, the sin(x), the logarithmic and the linear function. The parabola as well as the trigonometric function not restricted on a suitable interval of the domain are not injective on their domains, and the logarithmic as well as the linear function are injective on their whole domain. On lines 12-14, Zenobia explains to students visually on the graph why the parabola is injective on the interval $[0,\infty)$, and later on after this episode, she explains to them with a similar argument why the sin(x) is injective on [–

π/2,π/2]. Considering Mason and Pimm's (1984), Balacheff's (1988) and Rowland's (2002) emphasis on the presentation of the generic character of the example, here it seems that Zenobia presents to students how to find injective functions graphically and these four examples can thus be considered as generic of injectivity on intervals. The fact that Zenobia does not use the formal definition of an injective function to explain to students why the functions are injective on intervals indicates me that her examples are congruous with Mason and Pimm's consideration of generic examples. However, Mason and Pimm (1984, p.287) offer one example that carries the generality within it and suffices for its intended role as the "carrier of the general", whereas in this episode genericity is derived across four examples.

In my interview with Zenobia, she referred to the four examples as "special classes of functions", and she explained the reasons why she used them:

> Everything you see in polynomials is already seen in these two functions [i.e. the parabola, even degree, and the linear function, odd degree] so adding any additional polynomial you don't get anything new, whereas you never see periodicity or natural domain less than a whole axis in polynomials.

Her explanation in the interview reveals her view of the significance of the context of functions regarding the notion of injectivity. Looking at Mason and Pimm (1984), they stress that the example is generic in the context. In the case of function as opposed to elementary algebra, the notions are more complex since different kinds of functions have different properties and behaviour, so the context of functions is the one that imposes the need for dimensions of variation (Watson & Mason, 2005; Marton & Booth, 1997), i.e. the polynomial, logarithmic and trigonometric classes of functions. However, the four functions [lines 4-7] should have a high level of generality about them in order to be generic examples. The linear nature of the graph of $f(x)=x$ [line 7] indicates that it is not a generic example; more precisely, it is a prototypical example (Lakoff, 1987) of injectivity on an interval, since not all injective functions are linear. On the contrary, a generic example of injectivity on intervals that is an odd degree polynomial function is $f(x)=x^3$, since it also has the change of concavity. The parabola is a non-prototypical even degree polynomial function, which is generic of injectivity on intervals. The logarithmic function carries layers of generality of injectivity on the domain, because it has non-zero curvature as opposed to the linear function and its domain is $\Re$+. The trigonometric function is periodic and injective on intervals. It is finally true that the four examples concern only one variable functions and not more variable functions; however at that time of their studies, the students were not aware of functions of more variables.

## Conclusions

*Lecturer's use of generic examples* demonstrates a characteristic that widely emerged in analyses within the project, and so the need to show it in the data arose. Zenobia's use of generic examples reveals her effort to transpose the body of mathematical knowledge into forms that are accessible to students. So, the four examples serve as the didactical tool with which she explains and illuminates injectivity on intervals in her teaching. Devising the four examples was acting in the moment for Zenobia, which is different in nature from thinking in advance about what generic example (of the mathematical feature being taught) to use. So, the example of the linear function came on a spur of the moment decision and a 'better' example in terms of genericity (such as the polynomial function $f(x)=x^3$) might have been selected if it was a product of a long-considered thinking. The complexity of the mathematical concept being taught, in this case injectivity on intervals, and the complexity of the mathematical context of functions necessitate genericity across examples and not within an example. Mason and Pimm (1984) stress that when lecturers are not clear about the genericity of the example they use, students tend not to link it with the large class. Thinking of the case of a non-transparent presentation (Rowland, 2002) of genericity across cases, it might be even more difficult for students to see the generic role in a set of examples rather than a single case.

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## References

Balacheff, N. (1988). Aspects of proof in pupils' practice of school mathematics. In D. Pimm (Ed.), *Mathematics, Teachers and Children* (pp. 216-235). London: Hodder and Stoughton.

Jaworski, B. (2003). Sensitivity and challenge in university mathematics tutorial teaching. *Educational Studies in Mathematics*, *51*, 71-94.

Lakoff, G. (1987). *Women, Fire and Dangerous Things: What Categories Reveal about the Mind*. Chicago: University of Chicago Press.

Marton, F., & Booth, S. (1997). *Learning and awareness*. Mahwah, NJ: Lawrence Erlbaum Associates.

Mason, J., & Pimm, D. (1984). Generic examples: Seeing the general in the particular. *Educational Studies in Mathematics*, *15*, 277-289.

Nardi, E., Jaworski, B., & Hegedus, S. (2005). A Spectrum of Pedagogical Awareness for Undergraduate Mathematics: From "Tricks" to "Techniques". *Journal for Research in Mathematics Education*, *36*(4), 284-316.

Nardi, E. (2008). *Amongst Mathematicians: Teaching and Learning Mathematics at the University Level.* New York: Springer.

Petropoulou, G., Potari, D., & Zachariades, T. (2011). Inquiring mathematics teaching at the university level. In B. Ubuz, B (Ed.), *Proc. 35$^{th}$ Conf. of the Int. Group for the Psychology of Mathematics Education* (Vol.3, pp. 385-392). Bergen, Norway: PME.

Presmeg, N. C. (1992). Prototypes, metaphors, metonymies and imaginative rationality in high school mathematics. *Educational Studies in Mathematics, 23*(6), 595-610.

Rowland, T. (1999). 'i' is for induction. *Mathematics Teaching*, *167*, 23-27.

Rowland, T. (2002). Generic proofs in number theory. In S. R. Campbell & R. Zazkis (Eds.), *Learning and teaching number theory: Research in cognition and instruction* (pp. 157-183). Westport, CT: Ablex.

Speer, N. M., Smith III, J. P., & Horvath, A. (2010). Collegiate mathematics teaching: An unexamined practice. *The Journal of Mathematical Behavior*, *29*, 99- 114.

Watson, A., & Mason, J. (2005). *Mathematics as a constructive activity: learners generating examples.* Mahwah, New Jersey, USA: Lawrence Erlbaum Associates.