Absolute and relational representations: the challenge of Caleb Gattegno and Bob Davis

Alf Coles

University of Bristol

How do we learn mathematical concepts? How can we learn them fast? In this paper, I offer a lighting on the work of Gattegno and Davis and suggest that one common feature was a linking of mathematical concepts and symbols to relations (e.g., relations between physical objects, or actions performed on the objects). Visible and tangible resources are used to support the awareness of relationships *between* symbols, rather than offer a meaning *for* symbols. I suggest a distinction between an 'absolute' and a 'relational'representation of mathematical concepts *(I am endebteed to Tim Rowland for suggesting these labels at a BSRLM conference).*

Keywords: absolute representation, relational representation, mathematics teaching, symbols, fluency

Concepts and conceptual development

How can we make the learning of mathematics efficient (in terms of time) for the students we teach? Or, putting this another way, how do we learn mathematical concepts, and how do we learn them fast?

There has been considerable work, within mathematics education about how concepts are learnt. Table 1 is a summary (from Tall et al., 1999, p.4) of fifty years work about the way humans come to be able to operate with mathematical concepts.

	Process	…	Object
Piaget (50s)	action(s), operation(s) …	…	thematized object of thought.
Dienes (60s)	predicate …	…	subject.
Davis (80s)	visually moderated sequence … *each step prompts the next*	integrated sequence … *seen as a whole, and can be broken into sub-sequences*	a thing, an entity, a noun.
Greeno (80s)	procedure …	input to another procedure …	conceptual entity.
Dubinsky (80s)	action … *each step triggers the next*	interiorized process … *with conscious control*	encapsulated object.
Sfard (80s)	interiorized process … *process performed*	condensed process … *self-contained*	reified object.
Gray & Tall (90s)	procedure … *specific algorithm*	process … *conceived as a whole, irrespective of algorithm*	procept. *symbol evoking process or concept*

Table 1: The transition between process and object

There is a suggestion, across all these authors, that the movement in learning is from left to right across Table 1. A commonality, across these perspectives, is then the sense of starting with a process and ending with some kind of object that either contains the process as a 'thing' or allows a flexible evocation of object or process.

There is an implication, therefore, that teaching starts with a process and the question of how to make learning fast shifts to: what kinds of processes can we offer students that will make the transition to 'object' efficient?

To take a partial view of developments in mathematics education since the 1990s, I highlight two bodies of work. The first is the linguistic turn (Morgan, 2006; Barwell, 2009) which moves away from questions of cognition, to ask how learners talk about mathematics and to investigate how mathematical objects arise through discourse. Meaning is seen not as an internal event (with some students 'getting' the meaning and others inevitably not) but as arising through dialogue. The second movement is the increasing interest in the embodied nature of mathematical knowing (Lakoff and Nunez, 2000). One strand of thinking within this movement is the perspective of enactivism (Varela, Thompson and Rosch, 1991) in which meaning and significance are seen as arising through the interaction between a system and its environment. Knowing is equated to doing, and doing to knowing (Maturana and Varela, 1987). No matter what perspective is taken in relation to conceptualising and learning, however, there are paradoxes that have been recognised since Antiquity.

Problems in learning concepts

Plato raised the paradox of learning (Meno, 80d) that if learning is the recognition of the new, how is this ever possible, since to recognise something I need to know what I am looking for? Several centuries later, essentially the same paradox is linked to sign use by St Augustine. Signs, in this context, can be taken to include words.

> Now if we consider the matter more diligently, perhaps you will find that there is nothing that is learned by signs proper to it. For when a sign is presented to me, if it finds me ignorant of the reality of which it is a sign, it cannot teach me anything; but if it finds me knowing the reality, what do I learn by means of the sign? (St Augustine, 1978, p.173)

When a new word is presented to me, I either recognize what the word refers to or I do not (according to a classical view of language). If I do not recognize what the word is indicating, then what can the word on its own teach me? If I do recognize what the word refers to, then I learn nothing new by the addition of the word. St Augustine offers a way out of this paradox, through a focus on 'reality' rather than words.

> we learn the meaning of the word – that is, the signification that is hidden in the sound – only after the reality itself which is signified has been recognised, rather than perceive that reality by means of such signification. (ibid, p.174)

If we make a new distinction in the world, then we can attach a new word to it. But a new word, on its own, will not mean that we perceive the world differently.

More recently Anna Sfard, at a conference, referred in her own way to the same paradox again: 'To participate in a discourse on an object you need to have already constructed this object but the only way to construct an object is to participate in the discourse about it' (personal communication, 2013) (see Sfard, 2013).

In presenting these different versions of the paradox of learning, I have deliberately picked out the language of "having" and "gaining" new objects, new concepts or new understanding. Yet perhaps it is this "ownership" model that leads us into paradox in the first place. One of the problems of a focus on understanding mathematical concepts is that, to quote from Dick Tahta (1989):

> [t]his, then, inevitably leads to metaphors of ownership and control: obtaining the meaning, having the understanding, getting the concept. And, consequently, of course, there will be the mathematical descaminados, the shirtless who have not

> understood, who never get the concept … I always have some concept of what we may both be considering. I will certainly never have yours.

And once we conceptualise there being mathematical 'haves' and 'have nots' it is a short step to placing them in different 'ability groups'. In the UK, it would not be uncommon for 4 year-old children to be taught different mathematics, dependent on a judgement of their prior attainment. There is evidence that there is often little significant movement between 'ability' sets and that life chances are largely determined by mathematical achievement at aged 16, which is largely determined by your set. And of course, in the UK, the strongest predictor of being labeled one of the descaminados for your entire school career, is social class. It is, therefore, in part a question of social justice to investigate alternatives to a focus on 'understanding'.

An alternative vision

So, what would teaching look like if there was not a focus on developing student understanding of mathematical concepts? It might seem that the only alternative is rote learning, or 'traditional teaching'. Gattegno's own practice and that of Davis, I suggest, offers an alternative. Gattegno (1974) developed a curriculum, using which he claimed that the entire UK secondary mathematics curriculum (which takes most students the five years between ages 11 and 16) could be taught, to mastery, in eighteen months. Gattegno's concern was with the efficiency of learning and what, as teachers, we can do to minimize the time we take from students in, for example, the study of mathematics. Gattegno's vision for mathematics teaching is one where the teacher places *no* attention on encouraging 'understanding', but rather on supporting gains in symbolic mastery, letting students generate over time what meaning they will, for what they are doing. In Dick Tahta's (1989) pithy summary: 'the teacher looks after the symbols. The sense looks after itself'.

What Gattegno does offer students are 'plenty of things to try out which give students something to do and something to talk about' (Tahta, 1989). One way out of the the paradox of learning is to get students operating with concepts, initially in a "part-whole" or metonymic manner. Rather than the assumption that students need to "understand" symbols (metaphorically) before they can use them, consideration can be given to setting up contexts, or structures, in which symbols can be introduced with a limited number of dimensions of variation (Marton and Booth, 1997; Mason, 2011), but yet with the capacity for these dimensions of variation to increase with no rupture to the original conception.

I will next draw on examples from two archive video recordings, one of Gattegno teaching and the other of Davis, in order to illustrate such contexts.

Classroom example – Bob Davis

Rutgers University, in the USA, is home to a unique video library of mathematics teaching (http://videomosaic.org/), spanning over 50 years and starting with a project instigated by Bob Davis (the 'Madison Project'). Part of this collection is some historic video footage of Bob Davis working with different groups of students.

In one clip he is teaching children who look to be around age 5. The clip begins with him at the front of the class with two children. One of the children, Paul, is holding a bag of stones. He asks the other student, Bruce, to say "Go" and then asks the class how many stones they would like him to put in the bag. A student suggests 5 stones, so Davis counts out 5 stones from his pocket and places them in the bag. He turns to the blackboard and writes "5", saying as he is doing this 'I better write that before I forget’. He asks ‘Are there more stones in the bag than when Bruce said ‘go’ or are there less?’ and he chooses Nora to tell him ‘More’, ‘And, how many more?’, ‘5’ she responds. He then asks the class how many stones they would like him to take out of the bag. A student suggests 6, another student asks, ‘Did you have stones in the bag to start with?’. Davis responds, ‘I better have had, hadn’t I, or I wouldn’t be able to do this’ and takes a handful of stones from the bag and counts them out on his palm, he had picked precisely 6, ‘More by good luck than good management’. He asks: ‘Have I got more stones now in the bag than when Bruce said ‘go’ or have I got less?’. He chooses Jeff, who says ‘Less’, ‘How many less?’, Nora says 1. Davis then completes the statement on the board, ‘$5 – 6 = {}^{-}1$’, a student tells him to write ‘negative one’ to indicate that the result is ‘one less’.

In this clip, we see Davis setting up a game-like situation with the class. What I want to draw out is the relationship between the actual stones and their representation with symbols. When Davis writes ‘5’ on the board, he is indicating that five stones have been added to the bag, the symbol therefore represents a change in the number of stones. The symbols do *not* represent the concrete objects in an absolute sense. Instead, the symbols represent *relationships* between the objects, or *actions* on those objects.

By associating symbols with relationships, the symbols (from the outset) are abstracted from objects. In the beginning of work with the symbols, the concrete can still be called upon and the initial dimensions of variation are limited. Yet, it is not hard to imagine that attention will soon turn to dealing with the symbols in their own right, evoking, if necessary, the actions they represent.

Also significant is that by associating number symbols with relationships, there is seemingly no difficulty in getting students working with negative numbers. When a symbol is a relationship, working with the inverse appears straightforward.

Classroom example – Caleb Gattegno

A second example comes from the teaching of Caleb Gattegno and another historic video clip, available at (http://www.calebgattegno.org/mathematics-at-your-fingertips.html). At 6.50 minutes through this clip, Gattegno starts getting students naming Cuisenaire rods (see Figure 1 for an image of some rods). He says ‘Let’s find some new names, if we call the white one *one*’ (up to this point the rods had been named by their colour). Students are then made to order their rods into a staircase and hold up rods of given names (and lengths). In one activity, Gattegno asks: ‘how many white rods in this’, holding up a rod (the class chant a response), ‘and this’, ‘and this’.

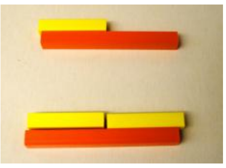

Figure 1: Relations with Cuisenaire rods

What I see as significant here is that number names are introduced in relation to a unit (the length of the white rod). The yellow rod does not represent ‘5’ (in an absolute sense), it represents the relation: ‘5 of the white rod’. But the yellow rod could also be called ‘1’, the orange rod then becomes ‘2’ (of the yellow), as in Figure 1.

Absolute and relational representations

The distinction between what I am calling an absolute and a relational representation is perhaps visible in considering the difference between introducing number names in a context such as the Gattegno video clip with links between the lengths of rods, and an introduction in the context of a resource such as ‘Numicon’ (see Figure 2).

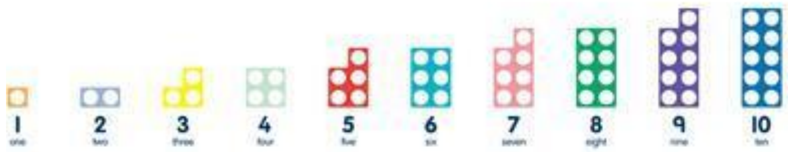

Figure 2: The "Numicon" number line

In the case of Figure 2, the manipulable objects are offered as *absolute* representations of number. Number '3' is the 'L' shape piece and it can be made, for example by putting the '2' and '1' piece together. The symbol '3' is used to represent just that particular object – hence the label 'absolute representation'. In contrast, Figure 1 is an alternative representation of the number '2' (if the yellow rod is called '1'). Here the symbol '2' points to a relationship between the yellow and orange rods. Two differences are pertinent. The first is that, using the Cuisenaire rods, alternative representations of '2' are possible (for example, 2 whites make the red rod). Secondly, it is apparent that the representation of '2' in Figure 1 can just as easily be seen as a representation of '½'. Of course, Cuisenaire rods can be used in exactly the same way as, say, Numicon, to represent specific numbers (in an absolute manner). The representation, via Cuisenaire rods, offers the potential of a relational view of number and, as seen in the Bob Davis video clip, when such a *relational* representation is used, there is little difficulty in considering the inverse.

The suggestion, then, arising from considering the teaching of Davis and Gattegno is that offering students relational (rather than absolute) representations of mathematics may be a powerful mechanism for making learning fast. With a relational representation of mathematical concepts, students can have powerful, simultaneous, access to inverse processes (negative numbers, with the Davis lesson; fractions, with the Gattegno lesson), which are typically seen as hard and left until later in the curriculum. There may also be advantage in terms of allowing students access to the ambiguity of mathematical symbolism. In Figure 1, the number 2 can be seen as the process of placing 2 rods against a bigger one, or as the object that results (see Coles (in press) for further elaboration of this point).

An implication for another part of the curriculum would be to introduce area not as an 'absolute' ("this shape has area 4") but to emphasise the unit ("this shape is 4 of that shape"). In turn, this suggests a possible re-ordering of the more usual topic sequence in mathematics. If a relational representation of the concept of 'area' draws attention to the role of the unit, then working on enlargement could be a *precursor* to working on area (and similarly for volume).

Relational representations of fractions would initially lead to an emphasis on their role as operators (rather than more absolute and static representations as, say, sectors of a circle). Work on angle would emphasise movement and comparison, work on symmetry and rotation might focus on geometrical transformations as operations. A relational view of functions might emphasise transformations of functions. A transformation approach to circle geometry is possible. In general, a relational representation of mathematical concepts draws attention to larger mathematical structures, in order to make teaching efficient. Symbols soon become meaningful in their connections to each other, and not linked directly to particular objects.

Conclusion

Gattegno and Davis were clearly gifted teachers and one counter-argument to the perspective in this article (that has been expressed to me) is that such teaching is not 'scalable', or is not possible for others to emulate. And here, perhaps, 'understanding' returns as an issue. One commonality between Gattegno and Davis is that they were both academic mathematicians at one point in their lives. Their understanding of mathematics was clearly deep and relational. Seeing Bob Davis work with young children on the distinction between an identity and an equation, is to see someone with a view of the long term development of mathematical thinking. Mathematics is offered as a coherent structure. There is an observable faith in students' sense making. Challenges are not made easier when they appear inaccessible to students.

One insight from the work, within teacher education, of Dick Tahta and Laurinda Brown is that role models of teaching are not there to be cloned. We can study in detail what, for example, Bob Davis was doing in his teaching, and the video clips give an image of what is possible with children, but if the aim is to support students in operating mathematically in a fluent manner, the parallel aim must be to encourage teachers to theorise their practice, in gaining their own fluency in the act of teaching. Gattegno and Davis offer an image of how efficient mathematics teaching can become. Tahta and Brown remind us that their challenge is surely not one of copying, but instead finding within ourselves what we need to work on in order to meet the needs of our students and colleagues.

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