Mathematical Modelling: Providing Valid Description or Lost in Translation

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With a focus on 'translation', we will discuss elements of a language of description that captures a range of strategies and criteria deployed in setting up a mathematical model and demonstrate its usefulness for analysing and evaluating school mathematical modelling activities. We see this as part of a larger project of exploring mathematical modelling and its recontextualisation in school mathematics.

Keywords: modelling, mathematisation, recontextualisation

Introduction

What is commonly referred to as mathematical modelling comprises a range of activities that often include chains of re-descriptions across professional practices and academic disciplines, and the activity might involve direct or indirect communication between experts from all of these. In school mathematics, there are two different approaches associated with mathematical modelling (see Jablonka & Gellert, 2011, for a discussion). One approach uses modelling as a pedagogic strategy for developing mathematics, as for example in *Realistic Mathematics Education*, where models are seen as vehicles to support 'progressive mathematisation' (e.g. Treffers & Goffree, 1985; van den Heuvel-Panhuizen, 2003). The other constructs modelling as a skill or competency to be acquired (e.g., Lesh & Doerr, 2003). Students are to be initiated into mathematical modelling by means of exemplary projects within the range of mathematical and other expertise of the participating students and teachers. In our discussion we shall refer to activities intended to serve this purpose.

In the mathematics education literature we find a variety of descriptions of 'the modelling process', which is in essence depicted as starting with some 'translation' into mathematics, sometimes referred to as 'mathematisation', and ending with 'validation', including cycles of adaption and refinement, or rejection. With a focus on 'translation', we present elements of a language of description that captures a range of strategies and criteria deployed in this stage of the process and demonstrate its usefulness for analysing and evaluating school mathematical modelling activities.

Background and Aims

Mathematical modelling overtly reads a non-mathematical practice from the viewpoint of mathematics and then reads the result from the viewpoint of that non-mathematical practice. It is this process of 'recontextualisation', which is central to our analysis of mathematical modelling activities. We rely on two schemes developed as part of Social Activity Method (SAM) (Dowling, 1998, 2009, 2013). A practice is recognised through its signs (ties between signifiers and signified) and to the extent that it is recognised its signs can be described as being strongly institutionalised. As there is a recognisable practice through strongly institutionalised expression and content (the *esoteric domain* of school mathematics), so there is an associated set of practices where the expression and content is weakly regulated with respect to the esoteric domain; the *public domain*. Given that the strengths of institutionalisation of expression and content are independent there are logically two further *domains: the expressive* and *descriptive* (see table 1 and also Dowling (1998, Chapter 6) and Dowling (2013, p.206) for detailed explanation).

	Content	
Expression	I+	I-
I+	*Esoteric Domain*	*Descriptive Domain*
I-	*Expressive Domain*	*Public Domain*

Table 1: Domains of Action

A ‘gaze’ is cast from the esoteric domain onto other practices, thus *selecting*, through the deployment of esoteric principles a public domain where such mundane activity as shopping is read *mathematically* as for, say, calculating the minimum unit cost (e.g. ‘best buy’ problems in mathematics textbooks).

Modelling is also to cast a gaze onto a non-mathematical practice; but here I+ expression articulates I- content. The result, initially, is a *recontextualisation* of the non-mathematical practice as descriptive domain text. This however does not capture the shifts in principles of evaluation involved in the modelling process mentioned above. That is, the modelling gaze is formed by mathematical principles and the gaze cast on the mathematical model is formed from the principles of the non-mathematical originating practice.

A further element of SAM, which proves useful here, is how far the principles of a practice are made explicitly available. This Dowling terms *discursive saturation*. A text which is highly discursively saturated makes available, explicitly, the principles of evaluation of the practice. Mathematics can produce such texts *par excellence* in the form of formal proof. The text aims at defining with near completeness the aspect of practice to which it refers. It might require an experienced reader to access the principle, but it still remains available. Other practices may be predominantly read as texts which are more weakly discursively saturated (DS-). In learning to ride a bicycle a novice might be given verbal instruction, but it is unlikely to prevent them falling off on their first few attempts. It is gaining a ‘feel’ rather than explicit principle which is more important at the outset.

The dimensioning of practices by their degree of discursive saturation provides the basis for an analysis of modes of recontextualisation (Dowling, 2013). The practice, casting a recontextualising gaze onto another, may be relatively DS+ or DS-. Similarly the practice being recontextualised may be DS+ or DS- (see Dowling, 2013, for an elaboration). As mathematics tends to make available its principles, mathematical modelling might be concerned with ‘rationalising’ (by casting a gaze onto a relatively DS- practice) or with ‘re-principling’ (by casting a gaze onto another DS+ practice). We find both versions in school mathematical modelling.

Our intention is to develop this schema in terms of *mathematical modelling modes of recontextualisation*. The development engages in a productive *re-working* of SAM (Dudley-Smith, 2014). As a starting point, we distinguish different strategies employed in the initial move of a modelling process, which is in the cycle descriptions commonly referred to as ‘mathematisation’.

Mathematisation Strategies

An analysis of a range of mathematical ‘modelling texts’ (such as examples and short descriptions developed for a non-expert audience, materials produced by teachers and students in the course of a modelling activity, outlines of modelling activities for classroom use) reveals a range of modelling activities, which employ different mathematisation strategies (Jablonka, 1996). As pointed out by Niss (1989, p. 27-28), one way of defining a mathematical model is to conceive of it as “the tripel (*A*,*M*,*f*), where *A* is the segment of reality under consideration, and f is a mapping which translates certain items of *A* into items of *M*”, where *M* is a “collection” of “mathematical objects, relations, structures, and so on”.

One main difference between mathematical models concerns the availability of a principle for deriving the mathematical relationships between variables, parameters, geometrical entities etc., signified in a mathematical model from the empirical. It has to be noted that such a principle relies on the possibility of some form of measurement. Another difference relates to the availability of theoretical principles, independent from any form of measurement, on which the mapping is based. Hence, *f* in the description by Niss would consist of two dimensions. For our analysis of modelling texts (and not ‘models as such’), we look at the availability of principles for these dimensions in terms of discursive saturation. This results in the scheme below (Table 2) which is a *reworking* of Dowling’s (2009, p.207) general scheme for ‘grammatical modes’.

Quantification rule (external syntax)	Mapping rule (internal syntax)	
	DS+	DS-
DS+	*Definitive Mathematisation*	*Ad-hoc Mathematisation*
DS-	*Derived Mathematisation*	*Originative Mathematisation*

Table 2: Mathematisation strategies

The point of this scheme is that it allows to capture differences in what is referred to as ‘translation’ or ‘mathematisation’ in the process description of mathematical modelling. Theoretically each of these strategies entails different strategies of model validation. This is an issue we intend to pursue further in our project of developing an empirically based language of description of modelling texts.

Examples of modelling texts that would resemble derived as well as originative mathematisation are discussed by Jablonka (2003) in the context of an elaboration of ‘mathematical literacy’ as interrogation of mathematical texts. Verhulst (1840) derived his well-known model for the growth of a population in the form of $P(n+1)=k\cdot P(n)-c\cdot P(n)^2$ exclusively from stated theoretical assumptions about the growth of a population that provide the basis for its mathematical structure. The (constant) parameter signifies the net proportionate excess of births over deaths and the quadratic term represents competition for resources. Hence, this is a *derived mathematisation*. *Originative mathematisation*, which leaves both the mapping and the quantification rule implicit, might not be considered as adhering to proper methodological standards, as noted by Richardson (1919) in his presentation of a ‘psychological model of war’:

> In this essay a very different use is made of mathematical symbols. […] each formula has been mentally compared with the miscellaneous facts known to the author, and the succeeding formula is often an improvement, a higher synthesis in the Hegelian sense, and not a deduction. [...] Indeed on account of the difficulty of defining the fundamental quantities, there remains a general vagueness, which may scandalise some of those who have been trained in the exact sciences, but which, in the author's opinion, does not deprive the formulae of meaning, interest and suggestiveness."
>
> (Richardson, 1919, p. 67, cited in Jablonka, 2003, p. 94)

Typical examples of *ad-hoc mathematisation* would be found in empirical statistics, and *definite mathematisation* in applications of classical mechanics.

Analysis of a Modelling Activity: ‘Keeping the Pizza Hot’ (The Bowland Trust)

The activity we analyse was produced in the context of the Bowland Maths Project (Burke, Hodgen & Olley, 2008). The problem (see Figure 1) was presented as a modelling activity. It dealt with the problem of cooling rates of pizza delivered in different forms of packaging. The problem was framed as a ‘real’ concern of a take-away pizza shop to increase its range for home deliveries, but being limited by the time a pizza can travel, on the back of a moped, without getting too cold. In the course of the activity an experiment was set up where a heated pizza was allowed to cool in different insulations. The temperature was recorded each second and the data were stored on a computer.

Our analysis of this modelling activity reveals a move from originative through ad-hoc mathematisation that results in a definitive mathematisation. Due to lack of space, we are not able to substantiate our analysis with detailed empirical data from the recorded classroom conversations. Hence, we present it in the form of a narrative that describes key changes in the mathematisation strategy.

The Pizza Delivery Problem

Make predictions

- After every reading predict the temperature at the **next** minute.
- After 2 minutes predict the temperature at 5 minutes.
- After 5 minutes predict the temperature at 10 minutes.
- After 10 minutes predict the temperature at 30 minutes, 120 minutes and 24 hours.

Time (T) (minutes)	Prediction Temperature (degrees C)	Actual Temperature (degrees C)	How did you make you prediction?
0			
1			
2			
3			
4			
5			
6			
7			
8			
9			
10			
30			
2 hours			
24 hours			

When will the pizza reach 48° ?

Figure 1: The pizza task (The Bowland Trust)

When faced with the experiment, many participants (including the teacher) 'knew' that things cool as an exponential law. This seemed plausible to them, but was neither validated nor explained beyond this. Hence they operated with a DS- mapping rule. Also, the initial temperature and estimates were 'guessed' and so the quantification rule was DS- from the outset. This means, the activity started with an originative mathematisation.

The apparent knowledge about an exponential law of cooling, however, was quickly discarded as the data unfolded. The quantification rule had some principles being imposed (e.g, it goes down by roughly the amount each time) and hence its discursive saturation was increasing. Pupils were able to select classes of functions from a fitting-by-eye basis to determine which would be the most appropriate to model the data. This informal perceptual relationship, phrased as, "It started at this point and goes down by this amount per unit interval", generated an initial algebraic form. The participants engaged in an ad-hoc mathematisation. Despite generating discursive saturation for the quantification rule, the empirical context generated critique. A strategy of hypothetical extrapolation of the initial function beyond the recorded data suggested that after a while the pizza would cool below 0º resulting in the concept of a 'self freezing pizza'. Other possibilities were considered. The "amount per unit interval" was varying by an amount per unit interval. But again, hypothetical extrapolation of a possible quadratic function, which was now considered as the alternative, would result in a 'self heating pizza'. Here the considerations moved to a theoretical level from which the relationships of the mathematical model could be derived. Knowing that there is a diminishing slope and a horizontal asymptote made the limitations of the ad-hoc mathematisation strategies apparent.

Aiming at increasing the discursive saturation of the mapping rule required some theory unavailable from the ad-hoc mathematisation of curve fitting. How would one determine 'sufficient' accuracy for validation? Why is the law of cooling exponential and not reciprocal? There was an emerging need for a scientific theory. Only theoretical considerations could move the activity towards a definitive mathematisation.

Conclusion

The analysis of the modelling activity shows the potential of exposing differences in mathematisation strategies, which are usually not captured in conceptualising modelling as sequential process, starting from the empirical. As we have pointed out, the mathematisation can be driven by a theory only (derived mathematisation), or start only from the empirical. In line with our terminology, the latter is commonly referred to as ad-hoc modelling, or not considered as modelling at all. We might find originative mathematisation as a strategy adopted by some pupils in contexts of 'emergent modelling', which Jablonka and Gellert (2011) described as operating on a covert recontextualisation principle. If modelling is to be included in mathematics classrooms, we consider it important to further develop a language of description with the potential to conceptualise the recontextualisation of mathematical modelling in school mathematics in order to provide a description that can be used for analysing, constructing and evaluating modelling activities, especially with respect to their possible stratification effects.

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