Algebraic reasoning in primary school: developing a framework of growth points.

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The purpose of my research is to explore to what extent children in Irish primary schools are developing skills in algebraic reasoning. In addressing the challenge faced by students encountering algebra in secondary school, one recommendation is to commence algebra instruction early in a child's schooling (Kaput, 2008; Carpenter and Levi, 1999; Mason, 2008; among others). Underpinning my research will be a framework of growth points, which outlines a possible developmental pathway in algebraic reasoning for primary school. In this paper, I present five growth points in algebraic reasoning with interim trajectories which may be supportive in exploring children's emerging understanding.

Keywords: Algebraic reasoning, developmental pathways, growth points, primary school.

Introduction

Algebra comprises one of the five strands of the Irish Primary Mathematics Curriculum (Government of Ireland, 1999). While children in Irish primary schools are engaged in mathematical activities under the algebra strand of the curriculum from their first year in primary school, an assessment of the mathematics attainment of 4000 pupils in Sixth class has shown that only 10% of those assessed could evaluate "linear expressions and one-step equations" (Eivers et al., 2010, p. 42). In my research I hope to investigate to what extent children are developing skills in algebraic reasoning as they progress through the Irish primary school system. To do so, I have developed a framework of growth points in algebraic reasoning, presented in this paper in a preliminary form. It is my intention within the framework that the growth points simulate large steps or significant advances in children's thinking. Drawing from literature, I have identified learning trajectories which may be associated with the framework of growth points. In this context, I am using the term 'learning trajectories' to describe developmental pathways which focus on small incremental advances but which do not have associated teaching activities. Using this framework and interim trajectories, I aim to assess children, through clinical interviews, to explore their emergent skills in algebraic reasoning.

Theoretical Perspective

In order to discuss how children develop skills of algebraic reasoning, it is pertinent to outline the theoretical stance I have taken with regard to what algebraic reasoning involves and also how developmental pathways may be envisaged. In this section I aim to foreground my conceptualisation of algebra and algebraic reasoning. I will also discuss theories of how children's development may be envisaged, focusing on hypothetical learning trajectories and growth points.

Algebra and algebraic reasoning

Attempts to define algebra often falter over the boundary between arithmetic and algebra. Radford (2012) asserts that "there is something inherently arithmetic in algebra and something inherently algebraic in arithmetic" (p. 676), but they are far from identical and it is useful to highlight what differentiates the two. For an activity to be considered algebraic, the child must make a leap from the exploration of the pattern to the identification of a generalisation (Radford, 2011). Similarly, for thinking to be algebraic, it must involve "*indeterminate* quantities conceived of in *analytic* ways" (Radford, 2011, p. 310, my emphasis). In considering what children need in order to engage with formal algebra, it is useful to focus on these skills of identifying a generalisation, and working with unknown quantities in analytic ways.

Kaput, Blanton and Moreno (2008) suggest that algebraic reasoning comprises the skills of "generalizing; expressing generalizations, and using specialized systems of symbols to reason with the generalizations" (p. 21). Algebraic reasoning relates to children's ability to think logically about quantities (known or unknown) and the relationships between them. Carpenter and Levi (1999) define algebraic reasoning by identifying two central themes, namely "making generalizations and […] using symbols to represent mathematical ideas and to represent and solve problems" (p. 2). The framework of growth points presented in this paper takes generalisation and symbol manipulation as two core skills of algebraic reasoning and considers how early algebra may facilitate children in developing such skills.

In developing a framework to plot a possible developmental pathway for algebraic reasoning, I will hold to the theory expressed by Mason (2008) and Hewitt (2009) that very young children possess skills which comprise algebraic reasoning. Research has shown that young children are capable of observing and applying some algebraic reasoning skills in mathematical contexts while not necessarily possessing the ability to express them as a general rule or to apply them to all situations (Schifter, Bastable, Russell, Seyferth and Riddle, 2008). Also, development of the framework is guided by an assumption that algebraic reasoning skills require intervention for their optimal development. Algebra is a cultural construct and children's skill development will benefit from facilitation by the children's educational environment (Radford, 2012; Kieran, 2011).

Developmental pathways

There are many and varied theories regarding the implications of tracking a developmental pathway. Rather than assuming a linear deterministic pathway where one minute learning goal follows from another when children are developmentally ready, the framework of growth points is designed to incorporate a non-linear trajectory. Such a trajectory allows for a range of progressions in children's learning, where some children may be expected to take a circuitous route to skill development, some may skip interim steps, and indeed some children may regress in order to surge forwards (Fosnot & Dolk, 2001). A learning trajectory designed to include all learners could not be linear or definite because until a child is presented with a concept, there is little way of being sure of how they will construct understanding (ibid.; Simon, 1995).

In developing a framework of growth points for algebraic reasoning in primary school, I would suggest that there is a necessity to frame a possible developmental pathway in broad growth points which incorporate many composite skills. In researching a framework of growth points for functions in early secondary school, Ronda (2004) identified “cognitive structures” or “meaningful chunks of information” as identifiable developmental progressions in children’s thinking. The growth points presented in this paper are based upon this definition, and I also aim to identify interim trajectories which may inform researchers and teachers about how under-developed or well-developed children’s skills are as they progress between growth points.

The framework of growth points

Incorporating the theories outlined above, a preliminary framework of five growth points in algebraic reasoning is presented in Table 1. Each growth-point is broadly envisaged and while the framework aims to track a developmental pathway for primary school, there are no age ranges or class levels automatically associated with growth-points. There are inconsistencies in research findings regarding the development of algebraic reasoning skills in Irish classrooms (Eivers et al., 2010; Government of Ireland, 2005), and I aim in the framework to allow for a broad variation in children’s prior engagement with algebraic reasoning.

Growth Point	**Characteristics**
GP 0: Pre-formal pattern	Children do not have a formal understanding of “pattern.” Children cannot identify a repeating term in a pattern.
GP 1: Informal pattern	Children can identify a commonality and demonstrate understanding of pattern by copying, extending, inputting missing term, in visual spatial, numeric, repeating and growing patterns.
GP 2: Formal pattern	Children can describe a pattern verbally. Children can offer a possible near (not next) term with reasoning.
GP 3: Generalisation	Children can correctly identify a near term. Children can describe a pattern explicitly. Children can offer a possible far term with reasoning.
GP 4: Abstract generalisation	Children can describe a pattern explicitly, describe the rule as an expression in symbolic notation and utilise the expression in order to generate a far term.

Table 1: A framework of growth points in algebraic reasoning.

The first three Growth Points are entitled Pre-formal Pattern, Informal Pattern and Formal Pattern. While there is no implication that the only skills relevant to these growth points are those of pattern solving, much discussion regarding extending algebra to the early years of schooling involves patterns and patterning. Lannin (2005) suggests that the emphasis on patterning as an introduction to algebra stems from the role of patterns as “dynamic representation of variables” (p. 233). The ability to generalise from patterns, and work with generalisations is fundamental to children’s developing thinking in mathematics (Warren and Cooper, 2008). Lannin (2005) suggests that generalizing through patterning activities may create a bridge between students’ knowledge of arithmetic and their understanding of symbolic representations.

Growth Point 0 and achieving Growth Point 1

On entering primary school, some young children would have already moved beyond GP0 but some may not. In the research available regarding the early algebraic activity of primary school children, there is much focus on patterning (Threlfall, 1999; Clements and Sarama, 2009; among others). There is a range of sub trajectories available in print, therefore, which outline the composite skills involved in early patterning, and the various approaches and strategies that children take (Rustigian, 1976; Vitz and Todd, 1969; Clements and Sarama, 2009). In my research I aim to integrate relevant interim trajectories from research which may be useful in monitoring children's progress towards GP1, Informal Pattern. In order to achieve GP1, children must have an understanding of pattern as growing or repeating, and also be able to consistently identify a commonality in a repeating pattern.

Achieving Growth Point 2, Formal Pattern

In order to achieve GP2 children must formalise their understanding of pattern and sequences. While the ability to consistently solve a pattern explicitly is not contained within GP2, children may be considering any relationships between terms and their positions. In achieving GP2, children must develop an ability to solve a pattern when the cognitive load implied by the pattern structure is increased. Warren and Cooper (2008) discussed the challenges encountered by children in solving a pattern involving two variables. Radford (2012) highlighted the difficulty encountered by children at this stage in linking numeric and spatial aspects of such a sequence. Incorporating the research of Warren and Cooper (2008) and Radford (2012), the interim trajectory which will suggest a developmental pathway between GP1 and GP2 will involve pattern solving where the cognitive load is gradually increased. The extent to which a child's skills have developed will be indicated by her success on increasingly challenging patterns.

Achieving Growth Point 3, Generalisation

When children begin to examine patterns, their natural reaction is to reason recursively, meaning that they examine the mathematical relationship between consecutive terms in a sequence (Lannin, 2004). As children develop sophistication in their approach to patterning, they need to move beyond the recursive method of exploring a sequence. To gain an insight into a greater range of patterning and also a structural understanding beyond the most basic repeating pattern, children need to develop an understanding that rules underlie patterns and that to expand a pattern efficiently, the rule must be identified. In order to achieve GP3 children must move from a recursive to an explicit approach to pattern solving. Lannin, Barker and Townsend (2006) outline a continuum of strategies which children may progress through as they develop the skill of explicit pattern solving, entitled Recursive, Chunking, Whole-Object and Explicit. In assessing children at this growth point who are not solving patterns explicitly, it may be useful to explore their strategy use as a means of informing the researcher as to their progression towards explicit pattern solving.

Achieving Growth Point 4, Abstract Generalisation

In achieving GP4 a student should demonstrate the range of skills outlined by Kieran (2004) as encompassed by an "algebraic way of thinking" (p. 140). Such skills include relational understanding of the equals sign, and the ability to work with problems and numerical expressions without feeling obliged to solve them. To achieve GP4, students should also be competent in dealing with abstract symbols, and be comfortable in "(i) working with letters that may at times be unknowns, variables, or parameters; (ii) accepting unclosed literal expressions as responses; (iii) comparing expressions for equivalence based on properties rather than on numerical evaluation" (Kieran, 2004, p. 140). To summarise, in achieving GP4 students will demonstrate competence in algebraic reasoning which will enable them to engage with abstract symbol systems and expression; and also to utilise algebraic notation and thinking in other areas of mathematics.

Conclusion

The framework of growth points presented in this paper is based upon a review of the relevant literature. It has not yet been piloted and is by no means all inclusive. In order to develop the assessment instrument required by my research, it is necessary at this point to identify relevant tasks and to script a clinical interview. In piloting the assessment, I will examine the assumed hierarchy of the growth points, accepting that while presented as linear, there is an understanding that not all children follow identical linear pathways in their learning.

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