Exploring the features of a collaborative connected classroom

Nicola Trubridge and Ted Graham
Plymouth University

This article considers the various dichotomies between types of mathematical understanding. It concludes that whilst the different categorisation is useful, it is the interplay and connections between these types of understanding that is more beneficial to student learning. Theories are drawn from a wide literature base to consider what this might look like in the secondary mathematics classroom and we propose the Collaborative Connected Classroom Model.

Keywords: making connections, collaboration, understanding.

Dichotomy of types of understanding

There are many different views of mathematical understanding and the debate has been on-going for decades. Often researchers and educators refer to a distinction between two types of understanding or knowledge. One of the first references to this is by Skemp where relational understanding is defined as "knowing both what to do and why" and instrumental understanding as "rules without reasons" (1976, p.2).

Byers and Herscovics (1977, p.26) were in agreement in principle of both relational and instrumental understanding but also put forward suggestions that there were some types of understanding that did not fall into either of the two categories. These were intuitive understanding, "the ability to solve a problem without prior analysis of the problem" and formal understanding, "the ability to connect mathematical symbolism and notation with relevant mathematical ideas and to combine these ideas into chains of logical reasoning".

Buxton (1978, p.36) rather than suggesting different types of understanding proposed four different levels of understanding. The first is rote which is the purely instrumental. The second is observational which is "slightly deeper than purely instrumental but not fully relational". The third level is insightful which is said to be relational. The fourth level, formal, refers to the definition from Byers and Herscovics which "is only appropriate after insightful or relational understanding is achieved and at a stage in the student's development where some idea of the need for and the nature of proof is accepted".

As well as the distinction into relational vs. instrumental understanding, the distinction of understanding into the categories of procedural and conceptual has been a focus of many research articles and is perhaps the most common.

> Conceptual knowledge is characterized most clearly as knowledge that is rich in relationships. It can be thought of as a connected web of knowledge, a network in which the linking relationships are as prominent as the discrete pieces of information … a unit of conceptual knowledge cannot be an isolated piece of information; by definition it is part of conceptual knowledge only if the holder recognises its relationship to other pieces of information. (Hiebert & Lefevre 1986, pp. 3-4)

It follows that constructing relationships between the pieces of information leads to the development of conceptual knowledge. These relationships are so important that Hiebert and Lefevre distinguish two levels at which these can occur, the primary and the reflective. At the primary level "the relationship connecting the information is constructed at the same level of abstractness ... than that at which the information is represented" (1986, p.4) whereas at the reflective level "the relationships transcend the level at which the knowledge currently is represented" (1986, p.5).

Hiebert and Lefevre divide procedural knowledge into two distinct parts "one part is composed of the formal language, or symbol representation system, of mathematics. The other part consists of the algorithms, or rules, for completing mathematical tasks" (1986, p.6). Despite the distinction into two types of procedural knowledge, they are similar in that all procedural knowledge relies on a "sequential nature" (1986, p.6). It appears to us that it is perhaps this sequential nature of relationships that makes it different from conceptual where the relationships can be of many different types.

The notion of 'connections' occurs regularly in the literature. Hiebert and Carpenter (1992) define conceptual knowledge so that it is identified with knowledge that is understood: "Conceptual knowledge is equated with connected networks" and procedural knowledge is defined as a sequence of actions (1992, p.78).

Interplay between types of understanding

Not only has there has been considerable debate about defining the two types of understanding, there has also been much debate as to which is the more important and which one should be taught first. Skemp (1976) acknowledges that even mathematicians who would classify themselves as relational still use instrumental thinking.

Byers and Herscovics (1977) assert that a good teacher can help a student to progress from intuitive understanding to formal understanding and similarly can support the move from instrumental to relational but that "the effective learning of mathematics cannot be based on one type of understanding. Nor ... can the different kinds of understanding be arranged in a linear order" (1977, p.27). They conclude that for optimal learning to happen the best approach is a spiral one so that "different types of understanding are used consecutively and repeatedly at even greater depth" (1977, p.27).

Hiebert and Lefevre (1986) acknowledge that the debate regarding two different types of knowledge has been ongoing for many years but recognise that the discussion has evolved over time and has moved from purely defining the types to looking at the relationships between them.

> Although it is possible to consider procedures without concepts, it is not so easy to imagine conceptual knowledge that is not linked with some procedures. This is due, in part, to the fact that procedures translate conceptual knowledge into something observable. Without procedures to access and act on the knowledge we would not know it was there. (Hiebert & Lefevre 1986, p.9)

Hiebert and Carpenter (1992) also acknowledge that both kinds of knowledge are required for mathematical expertise. They claim that uncovering relationships between conceptual and procedural knowledge is more useful than trying to establish which one is more important. Long (2005, p.61) claims that conceptual knowledge is intricately linked with procedures and algorithms. In fact, "knowledge of procedures is nested in conceptual knowledge".

Rittle-Johnson and Alabali (1999, p.188) propose that “conceptual and procedural knowledge appear to develop iteratively, with gains in one type of knowledge leading to gains in another”. However Askew et al (2010, p.34) state that “procedural fluency and conceptual understanding are largely seen as mutually exclusive aims”. Advisory Committee on Mathematics Education (2011, p.1), recognise the importance of procedures but they also acknowledge “for mathematical proficiency, learners need to develop procedural, conceptual and utilitarian aspects of mathematics together”.

Collaborative Connected Classroom Model

The concluding theme is that neither procedural/instrumental nor conceptual/relational knowledge is more important. They should be taught together with the importance on making connections. We have developed the Collaborative Connected Classroom Model after reviewing studies where there was a focus on classrooms that are connected in nature. These ideas have been synthesised and classified into four domains detailed below.

Teachers' Beliefs about Mathematics and Learning

The important overriding theme, that is consistent throughout the literature (Skemp, 1976; Askew et al., 1997; Swan, 2005; ACME, 2011), is that mathematics (is a subject that) contains a wide range of connections. These connections can be between different areas of mathematics (for example the use of proportional reasoning within the topics of similar triangles and conversions) and also between different representations (for example seeing an arithmetic sequence represented in its numerical, graphical and mapping forms).

Learning consists of building a conceptual structure (Skemp, 1976) and is a collaborative activity in which learners are challenged and arrive at understanding through discussion (Swan, 2005). The nature of this collaborative activity is detailed below, within the social culture of the classroom section.

The theory suggests that, within mathematics, mistakes are an important part of the learning process (Hiebert et al., 1997) and that they should be made explicit within lessons and developed as part of the lesson (Askew et al., 1997; Swan, 2005). These mistakes can provide essential opportunities to reconceptualise a problem (Kazemi & Stipek, 2008).

Nature of Mathematical Activity

If teachers believe that mathematics is ‘connected’ then teaching is about making learners engage with these connections. Mathematical activity will involve connecting different areas of mathematics or connecting different ideas in the same area of mathematics by making opportunities for a variety of words, symbols, diagrams and concrete situations (Askew et al., 1997; Haylock, 1982; Hiebert et al., 1997).

Whilst the notion of mathematical connections is not a new idea, the Collaborative Connected Classroom model aims to make more explicit the nature of the interplay of the connections between procedures and concepts. With this in mind mathematical tasks may make specific links between procedural and conceptual knowledge (Kadijevich & Haapasalo, 2001).

Tasks may take the ‘educational approach’ where meaning is built for procedural knowledge before mastering it (Kadijevich & Haapasalo, 2001), for example learners are encouraged to invent their own strategies before learning traditional algorithms. Or they may take the ‘developmental approach’ where procedural knowledge is used and then reflected on (Kadijevich & Haapasalo, 2001), for example comparison tasks are used where teachers encourage connections between the procedures being used to make generalisations resulting in conceptual understanding (Peled & Segalis, 2005).

Whichever approach (developmental/educational) is taken; mathematical tasks need to be accessible (Hiebert et al., 1997) and build on the knowledge that learners have (Swan, 2005) by connecting ideas to their current conceptual schema (Skemp, 1976). Misunderstandings should be made explicit so students can learn from them (Askew et al., 1997; Hiebert et al., 1997; Swan, 2005).

It is important that mathematical tasks are problematic and that application should be approached by challenges that need to be reasoned about (Hiebert et al., 1997; Askew et al., 1997). One method is that the teacher presents problems before explanations are offered (Swan, 2005) and they share essential information after selecting tasks with a goal in mind (Hiebert et al., 1997).

The Social Culture of the Classroom

There are many features apparent in a classroom where there is a focus on developing a social culture of more connected teaching. Research shows there will be a high degree of focussed discussion between teacher and whole class, teacher and groups of pupils, teachers and individual pupils and pupil themselves (Askew et al., 1997).

It is acknowledged that learners should emphasis methods rather than answers (Swan, 2005). However in the Collaborative Connected Classroom model the importance is on enabling learners to examine the mathematical similarities and differences between multiple strategies (Askew et al., 1997; Kazemi & Stipek, 2008). Teachers will work actively with the pupils’ explanations, refining them and drawing pupils’ attention to differences between methods (Askew et al., 1997; Kazemi & Stipek, 2008).

The important feature is that all learners will be encouraged to contribute and share their methods where they justify their strategies mathematically – not simply a procedural description (Kazemi & Stipek, 2008).

There will be a strong emphasis on developing methods, reasoning and justification (Askew et al., 1997). Learners will be each held accountable and consensus should be reached through mathematical argumentation (Hiebert et al., 1997; Kazemi & Stipek, 2008).

Characteristics of Learners

In a classroom where there is a focus on developing a more connected understanding of mathematics, learners will use strategies, which are both efficient and effective (Askew et al., 1997). They will know what to do and why they are doing it (Skemp, 1976) as they will be fluent with connections in mathematics (ACME, 2011). As a result they will be more confident in looking at new problems and attempting them without outside help (Skemp, 1976).

Learners who have made connections between procedures and their underpinning concepts will know a range of concepts, symbols and procedures and how they are related (Hiebert & Lefevre, 1986).

The table below summarises our model of the Collaborative Connected Classroom.

Table 1: Collaborative Connected Classroom Model

Teachers Beliefs about Mathematics and Learning	• Mathematics is a highly interconnected body of ideas that involves understanding and reasoning about concepts and the relationships between them • Mistakes should be recognised and made explicit. They are opportunities to reconceptualise a problem explore strategies and try out alternative strategies • Learning consists of building a conceptual structure whereby ideas are revisited and extended • Learning is a collaborative activity where learners are challenged to arrive at understanding through discussion

Nature of Mathematical Activity	• Builds on the knowledge that learners bring by connecting ideas to their current conceptual schema.
	• Tasks either connect together different areas of mathematics or connect different ideas in the same area using different representations (symbols, words, diagrams)
	• Links are made between procedures and concepts
	    o meaning is built for procedural knowledge before mastering it ('educational approach')
	    o procedures are evaluated to promote conceptual understanding ('developmental approach')
	• Tasks involve comparisons; this may be looking for similarities or differences between ideas or looking at efficiency of method
	• Application tasks are presented as challenges that may be problematic and need to be reasoned about

Social Culture of the Classroom	• Ideas and methods are valued and each student is held accountable for thinking through the mathematics in a problem until a consensus is reached. • There is an emphasis on reasoning and justification and not simply giving a procedural description • High degree of focussed non-linear discussion between teacher and groups of pupils, teachers and individual learners and between learners themselves • Discussion involves examining mathematical similarities/differences/connections among multiple strategies and refining learners explanations
Characteristics of Learners	• Know what to do and why they are doing it • Know a range of concepts, symbols and procedures and how they are related. • Use strategies which are both efficient and effective • Are aware of connections within mathematics • Are confident in tackling unfamiliar problems

Concluding comments

In conclusion we have drawn together a range of research literature and acknowledge that whilst it is useful to explore the nature of procedural vs. conceptual or instrumental vs. relational knowledge, it is the interplay between these that might lead to gains in student learning. We have proposed a model that acknowledges the importance of making connections and have considered what this might like look like within the mathematics classroom. We plan to research further how this model could be implemented by collaborating with teachers as part of a professional development programme.

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