Diagrams in the teaching and learning of geometry: some results and ideas for future research

Keith Jones
School of Education, University of Southampton, UK

Diagrams are generally taken to be an integral component of doing and understanding mathematics. In the teaching and learning of geometry, the use of diagrams is not only because of the nature of geometrical objects, but also because a diagram is often a particularly effective problem representation that enables complex geometric processes and structures to be represented holistically. At the same time, learners can be misled by diagrams. This brief paper provides some results from research on the affordances and limitations of diagrams in the teaching and learning of geometry. The paper concludes by suggesting some ideas for future research.

Keywords: geometry, diagrams, teaching, learning

Introduction

Diagrams are graphical forms of conveying information. Such representations are frequently used by mathematicians (e.g. Samkoff, Lai & Weber, 2012) and in the teaching and learning of mathematics (e.g. Stylianou, 2002). There are many reasons why the use of diagrams is so widespread in mathematics and its teaching and learning. Some of these reasons are captured in the expression ‘a picture is worth a thousand words’ or the equivalent ‘hearing a hundred times is not as good as seeing once’ (the latter being attributed to the *Book of Han*, completed in China in 111 CE).

In mathematics and its teaching, diagrams come in many different forms. Static mathematical diagrams appear in books and journals, while dynamic and interactive diagrams make use of the capabilities of digital technologies (for examples of the latter, see Yerushalmy & Naftaliev, 2011).

Considering diagrams in general, Winn (1987) identified a difference between graphic forms (such as charts and graphs) and diagrams. In this view, the function of graphs and charts is to show relationships between variables. In contrast, according to Winn, the function of diagrams is to “describe whole processes and structures often at levels of great complexity” (p. 153). This is the function that is at the heart of this paper.

The aim of this brief paper is to illustrate some results from my recent research on the affordances and limitations of diagrams in the teaching and learning of geometry. While some of my research has been on dynamic and interactive diagrams in geometry education (e.g. Jones, 2000; Jones, 2011), my focus for this paper is some recent research on static diagrams in geometry education. I conclude with some suggestions for future research on geometrical diagrams. Before doing so, I say a little more about the uses of diagrams in mathematics and its teaching and learning.

Diagrams in mathematics and mathematics education

As an example of the uses of diagrams in mathematics by mathematicians, it is worth examining the study by Samkoff et al. (2012) noted above. In this study, the authors presented eight mathematicians with a task that invited the construction of a diagram. The analysis of the data focused on how the mathematicians used their diagram to produce a formal proof. The main findings were that the mathematicians "varied in the extent of their diagram usage" (p. 49). While the researchers noted that "it was not trivial for participants to translate an intuitive argument into a formal proof", they found that the mathematicians' reasons for using diagrams included "noticing mathematical properties, verifying logical deductions, representing ideas or assertions, and suggesting proof approaches" (ibid).

Such use of diagrams by mathematics suggests that diagrams might be used in related ways in mathematics education. As Samkoff et al. (2012) say "diagrams are viewed by mathematicians and mathematics educators alike as an integral component of doing and understanding mathematics" (p. 49); what is more "drawing diagrams is commonly cited as a heuristic for mathematical problem solving that students should engage in" (p. 50).

Yet, as Samkoff et al. also note, the processes involved in using diagrams are "surprisingly complex" (ibid). This echoes the more general findings of Larkin and Simon (1987) that a diagram is "(sometimes) worth ten thousand words" (to quote the title of their well-known paper). In terms of mathematics education, Schoenfeld (1985) is one who has argue that more explicit teaching that focuses on the uses of diagrams (for example, what to read from diagrams) might be helpful for learners of mathematics.

Diagrams in geometry education

One component of mathematics education that makes great use of diagrams is the teaching and learning of geometry (see, for example, chapter 5 of Watson, Jones & Pratt, 2013). Following the entry in the Oxford English Dictionary (nd) for 'geometrical diagram', the approach I take in this paper is that a geometric diagram is "a figure composed of lines, serving to illustrate a definition or statement, or to aid in the proof of a proposition".

While being composed of lines that serve to illustrate a definition or statement, or aid in the proof of a proposition, a geometrical diagram is generally accompanied by symbols and words (notwithstanding the idea of 'proof without words'; see Nelsen, 1993). Some of the symbols and associated words are illustrated in Table 1.

The three uses of geometrical diagrams that are the focus of the remainder of this paper are diagrams in school mathematics textbooks, diagrams in student geometrical problem solving at school, and diagram use by school teachers when teaching geometry.

Diagrams in school mathematics textbooks

Geometrical diagrams appear widely in the geometry sections of school mathematics textbooks. In a recent study of the geometry sections of school mathematics textbooks reported in Jones and Fujita (2013), the pages of the textbooks were divided into 'blocks' through use of the TIMSS categorisation of 'central instructional narrative', 'related graphic', worked examples with diagrams, exercise set with diagrams', and so on. While the overall focus of the analysis reported in the paper was how National

Curricula for mathematics are interpreted in school mathematics textbooks, the analysis did reveal some data on the prevalence and distribution of geometrical diagrams in the geometry sections of the textbooks. The relevant data is shown in Table 2.

Common Symbols Used in Geometry

Symbols save time and space when writing. Here are the most common geometrical symbols:

Symbol	Meaning	Example	In Words
Δ	Triangle	ΔABC has 3 equal sides	*Triangle ABC has three equal sides*
$\angle$	Angle	$\angle$ABC is 45°	*The angle formed by ABC is 45 degrees.*
$\perp$	Perpendicular	AB $\perp$ CD	*The line AB is perpendicular to line CD*
$\parallel$	Parallel	EF $\parallel$ GH	*The line EF is parallel to line GH*
°	Degrees	360° makes a full circle	
∟	Right Angle (90°)	∟ is 90°	*A right angle is 90 degrees*
$\overline{AB}$	Line Segment "AB"	$\overline{AB}$	*The line between A and B*
$\overleftrightarrow{AB}$	Line "AB"	$\overleftrightarrow{AB}$	*The infinite line that includes A and B*
$\overrightarrow{AB}$	Ray "AB"	$\overrightarrow{AB}$	*The line that starts at A, goes through B and continues on*
$\cong$	Congruent (same shape and size)	ΔABC $\cong$ ΔDEF	*Triangle ABC is congruent to triangle DEF*
$\sim$	Similar (same shape, different size)	ΔDEF $\sim$ ΔMNO	*Triangle DEF is similar to triangle MNO*
$\therefore$	Therefore	$a=b \therefore b=a$	*a equals b, therefore b equals a*

Table 1 Symbols and associated words that accompany geometrical diagrams

	Grade 8 (Year 9) textbook from Japan	**Grade 8 (Year 9) textbook from England**
Number of 'lessons' on geometry	34 'lessons' (out of 93 'lessons' in total for the year)	33 'lessons' (out of 121 'lessons' in total for the year)
Number of 'blocks' across the lessons	321 'blocks'	174 'blocks'
Proportion of 'Related graphic' blocks	14% of blocks	12.1% of blocks
Proportion of 'Worked examples with diagrams' blocks	8.4%	12.6%
Proportion of 'Exercise set with diagrams' blocks	16.2%	19.0%
Use of graphic or diagram	38.6% of blocks	43.7% of blocks

Table 2 The prevalence of geometrical diagrams in school mathematics textbooks for Grade 8 (Year 9)

The data in Table 2 suggests that while geometry comprises a smaller proportion of the lessons in the respective textbooks, for the book from England there was generally a higher rate of use of geometrical diagrams compared with the equivalent textbook from Japan. The research that generated this data did not focus in detail on the prevalence of diagrams in the geometry sections of the textbooks. More research on this issue would be valuable.

Diagrams in student geometrical problem solving at school

In a recent classroom teaching experiment, Jones, Fujita and Kunimune (2012) used the geometry task with the diagram in Figure 1 with Grade 8 (Year 9) students.

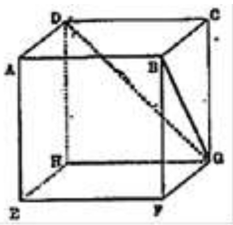	What is the size of the angle BGD? State your reason why.

Figure 1 Angle in a cube problem

Here a static diagram was deliberately chosen in order to focus on students' capabilities with such a diagram. In the first phase of the teaching experiment lesson, 28 students (out of 46) considered that angle BGD would be 60 degrees, three said 90 degrees, and 15 said 'I am not sure'. After ideas were shared amongst the class, three of the unsure students opted for the answer of 60 degrees, making a total of 31 students (ie 67%) conjecturing that the angle was 60 degrees. This meant that one-third of the students were not sure that the angle would be 60 degrees.

After further discussion, one student who was sure that angle BGD would be 60 degrees suggested using diagrams that have alternative orientations of the cube. These alternative diagrams helped to convince the class that triangle BGD is indeed equilateral. It was a suitable diagram, alongside suitable words and symbols, which helped to convince the doubting students. More research is needed on the relationships between diagrams, words and symbols in students' geometrical problem solving.

Diagram use by school teachers when teaching geometry

In a recent study of diagram use by school teachers when teaching geometry, Ding, Jones & Zhang (2013) report on the teaching approach of expert mathematics teachers in Shanghai, China. In a case study, the authors show how the teaching made use of a set of strategies that have been called '*Shen Tou*', or the 'permeation method'. This method comprises four teaching strategies that are used in combination:

- The Word-Symbol strategy: translating word language of concepts, definitions and proposition into geometrical symbol language with diagrams
- The Read-Draw and Oral-Draw strategy: reading word statements and then drawing the diagram (R-D), and drawing the diagram by listening to the teacher's oral statements (O-D)
- The Observation-Talk strategy: talking about the properties of the diagram when observing a diagram (O-T)

- The Word-Diagram-Symbol strategy: Based on the O-T, to use word language correctly and as concisely as possible to generalize geometrical facts (e.g., propositions and theorems) according to the diagram and symbol language (W-D-S)

Overall, the study by Ding, Jones & Zhang illustrates how an experienced and expert teacher uses the 'Shen Tou' method gradually to develop the multiple layers of reasoning skills required in geometry, especially the skills to use geometrical language in writing proofs. More research is needed on diagram use by school teachers when teaching geometry.

Concluding comment

Diagrams are in widespread use in mathematics and its teaching and learning. This is especially the case in geometry and in geometry education. Such diagrams can be static (as in regular books and journals) or dynamic and interactive (when utilising digital technologies). This particular paper has focused on static diagrams as these remain in common use. The three uses of such geometrical diagrams that have been the focus of the bulk of the paper are as follows: diagrams in school mathematics textbooks, diagrams in student geometrical problem solving at school, and diagram use by school teachers when teaching geometry. In each case, recent research which illustrates some of the issues with geometrical diagrams has been summarised. In each case, much remains unclear or unknown about the impact and use of geometrical diagrams. It is clear that diagrams are invaluable in aiding the teaching and learning processes; it is also clear that the processes involved in using diagrams are surprisingly complex (to use the phrase of Samkoff et al). All this points to the need for more research into diagrams in the teaching and learning of geometry.

References

Ding, L., Jones, K, & Zhang, D. (2013). Teaching geometrical theorems in Grade 8 using the 'Shen Tou' method: a case study in Shanghai. In L. Fan et al. (Eds), *How Chinese Teach Mathematics*. Singapore: World Scientific.

Jones, K. (2000). Providing a foundation for deductive reasoning: students' interpretations when using dynamic geometry software and their evolving mathematical explanations, *Educational Studies in Mathematics, 44* (1-3), 55-85.

——— (2011). The value of learning geometry with ICT: lessons from innovative educational research. In Adrian Oldknow and Carol Knights (Eds.) *Mathematics Education with Digital Technology* (pp. 39-45). London: Continuum.

Jones, K. & Fujita, T. (2013) Interpretations of National Curricula: the case of geometry in textbooks from England and Japan. *ZDM: The International Journal on Mathematics Education, 45 (5)*, pp. tbc.

Jones, K., Fujita, T. & Kunimune, S. (2012). Representations and reasoning in 3-D geometry in lower secondary school. In T-Y Tso (Ed.), *Proceedings of PME-36* (Vol. 2, pp. 339-346). Taipei, Taiwan: PME.

Larkin, J. H. & Simon, H. A. (1987). Why a diagram is (sometimes) worth ten thousand words. *Cognitive Science, 11,* 65-69.

Nelsen, R. B. (1993). *Proofs without Words: Exercises in visual thinking*. Washington, DC: Mathematical Association of America.

Oxford English Dictionary (nd). www.oed.com

Samkoff, A., Lai, Y. & Weber, K. (2012). On the different ways that mathematicians use diagrams in proof construction, *Research in Mathematics Education, 14* (1), 49-67.

Schoenfeld, A. (1985). *Mathematical Problem Solving*. Orlando, FL: Academic Press.

Stylianou, D.A. (2002). On the interaction of visualization and analysis: The negotiation of a visual representation in problem solving. *Journal of Mathematical Behavior, 21* (3), 303-317.

Yerushalmy, M. & Naftaliev, E. (2011). Design of interactive diagrams structured upon generic animations. *Technology, Knowledge and Learning, 16* (3), 221-245.

Watson, A., Jones, K. & Pratt, D. (2013). *Key Ideas in Teaching Mathematics: research-based guidance for ages 9-19*. Oxford: Oxford University Press.

Winn, W. (1987). Charts, graphs, and diagrams in educational materials. In D. M. Willows and H. A. Houghton (Eds.), *The Psychology of Illustration. Volume 1 Basic Research*. New York: Springer.