The study of intuitions in one prospective teacher's constructions of mathematical objects

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The research is a study of the Husserlian approach to intuition, as it is substantiated by Hintikka, in the case of a prospective teacher of mathematics. It is a case study based on data collected from a course where the students were engaged in open-ended tasks and they were free to choose their own ways of exploring them while working in groups, without the teacher's intervention. A phenomenological approach that takes objects as self-given and focuses on the student's intuitions reveals mathematical objects that surfaced from her investigation and the particular circumstances that led to these objects. The research exemplifies the two intuitive stages introduced by Husserl, while introducing a method of discerning them, and argues for the essential part that intuitions play in the construction of mathematical objects.

Keywords: empirical intuitions, abstract intuitions or intuitions of essences.

Introduction and theoretical framework

The concept of intuition is stressed as an important feature of mathematical learning activity and the bibliography on this topic is particularly rich (Beth and Piaget 1966; Fischbein 1994; Hersh 1997; Poincare 2007). Most approaches of intuition are using a *constructivist* lens, originating in Kant, Descartes or Plato. Although there is not enough space here to explore these issues, the general idea is that processes that inhabit the *cogito*, the discursive human mind take over, and the abstract intuitive processes are ultimately settled in internal, relational cognitive models. The world from which intuition derives its materials is either internal (in the case of abstract objects) or external (in the case of empirical objects). For these approaches intuition is a mental act that presents empirical or mental objects to us, based on pictorial models, ideality, abstractness, absolute perfection and universality (Fischbein, 1994).

What is common in most approaches to intuitive thinking is that they do not give a clear picture of what intuition actually is and how it emerges. Hersh (1997, 65) maintains that "[w]e have intuition because we have mental representations of mathematical objects. … We don't know how these representations are held in the mind/brain. We don't know how any thought or knowledge is held in the mind/brain". Fischbein (1994, 88) explains that when a child "affirms that a line may be extended indefinitely, he expresses an intuition. This intuition is *related to his experience*" (italics added). Fischbein's claim about *behavioural roots of intuitive representations* (ibid, ch. 7) is concomitant to the approach of constructivism and its radical ramifications to perception in general (Cobb 1994; Cobb and Yackel 1996; Steffe and Kieren 1994; von Glasersfeld 1995), being faithful to the Kantian perception of experience as knowledge:

> When Kant maintains that the "I" makes experience possible, he is working with a very special concept of experience. By experience Kant means *objective* thinking, *knowledge*. "Experience is an empirical knowledge," writes Kant (CPR 208 [B

> 218]). Hence for Kant experience is knowledge and already involves objective categories. (Tito 1990, 78, italics in the original).

The radical constructivist turn only bracketed the ontological theoretical commitments to the Piagetian cognitive pedagogy, by classifying them as *trivial* (Steffe and Kieren 1994, 720-721), intending to release the possibilities for expansion of the Piagetian claim for cognition through action. Although this is a necessarily rough description of a rich movement that broke from behaviourism, empiricism and naïve idealism (cf. Steffe and Kieren 1994), it may suffice for the space available to summarise by adding that constructivism never broke its ties with Kantian transcendental idealism, as it is in the core of its perception of individual cognition. In contradistinction to the Kantian approach the current study adopts a Husserlian perspective, where "there is a level of experience that has not yet been subjected to the objective categories, a level of experience that is the ground of the objective categories" (Tito 1990, 78). This level of experience and the *intuitions* that operate in it are the field and the target (respectively) of the data analysis of this research. Intuition is the essential mediator between the learner's world-as-lived and her objectification process. In Husserlian terms:

> What is immediately given to me will then at the same time be part of the mind-independent reality and an element of my consciousness. There has to be an actual interface or overlap of my consciousness and reality. (Hintikka 1995, 82)

Thus, intuition is seen as a channel of the "overlap of my consciousness and reality" and it has two principal features emanating from Husserl's theory and a third one accompanying the second feature, which allow us to *trace* and *classify* them as such:

- Intentionality, namely the directedness of consciousness towards objects, which is for Husserl a fundamental attribute of all conscious acts, especially in their 'pregnant' state (Husserl 1980, 106), as intuition certainly is. Intentionality is the property that makes intuition an objectifying act, for empirical and—most importantly—for abstract objects: "what makes seeing an essence an intuition is not that it is seeing an essence, but that it is seeing an object which is 'itself given'" (Hintikka 2003, 181).
- *Immediacy* is the intuitions' second critical feature, the one that distinguishes them from other concepts such as imagination (Hintikka 2003, 175, 176), and it comes with the *feeling of certainty.*

Under the light of the aforementioned theory, and using a phenomenological method for the collection of data and the analysis of a learning episode, concerning a prospective teacher, the research exemplifies the emergence and the development of empirical and abstract objects due to empirical and abstract intuitions respectively. The paper describes her path from her alienation during the activity, the use of the material she had available, the synthesis of the material to a method, and the implications that it had for her approach of teaching and learning mathematics. The data sources I used included the audio recording of an interview (1h 5min), written artifacts based on her homework assignment, and participant observation during the initial activities.

The course and the 'doubling modulo' activity

The research is a **case study** from a course that took place in the academic year 2010-2011 and involved a group of 13 students, training to be teachers in British secondary schools. It consisted of twenty 3-hour sessions, focusing on a variety of mathematical topics in which one of the main targets was the production of generalisations by the students. The students were working in small groups of 3 to 5 people of their own choice and they were suggested to continue their investigations at home and take them as far as they could. The students were interacting with other members of their group, as well as the other groups. Three or four of each student's favourite investigations were submitted and assessed at the end of the course. The assessment was a function of the students' advancement of the investigations, their individual perspective on them and the development of reflective capability of their own understandings. I was invited by the teacher to the sessions and the students also gave me permission to participate and collect data from their activities.

The task that will be analysed was called the 'doubling modulo'. The teacher would think of a certain number and he would write a smaller number on the board, double it, and write the new result on the board, next to the previous one, if the result was smaller than the initial number. He continued this process until the doubled number would exceed (or become equal to) the fixed number. In this case the teacher subtracted the fixed number from the doubling number and the doubling process continued with the residue, the 'modulo', until he would find zero or a number that he had already used (a loop). He then started the same process again with any unused number smaller than the fixed one, until all numbers smaller than the fixed number would be recycled. The only instruction the students received by the teacher was to pursue the strongest generalisations possible. The students knew that the teacher would not respond to questions concerning the treatment of the task, and it was not always a case of a more or less smooth self-organisation or mutual adaptation in meanings negotiation and construction between the students. The teacher only wrote the first two or three numbers and then he was waiting for the students' response, writing the next number either when they got it right or after they couldn't find it. He expected the students to understand and share their understandings only by the repetition of the process. As soon as some of the students had an idea of what the teacher was doing they checked it and started sharing it with the other students. Their different strategies were related to the exploration of the natural numbers' properties (i.e. even, odd, prime numbers, powers of 2, prime factorization etc.) and how these properties were related to the results (see Figures 1, 2).

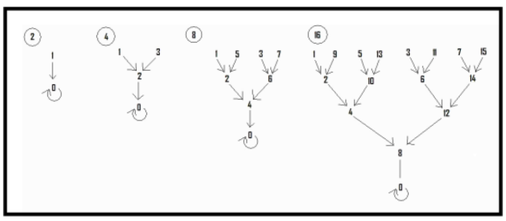

Figure 1

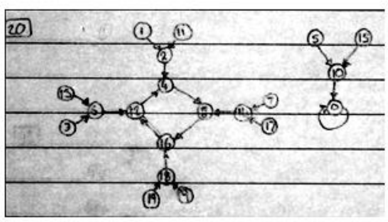

Figure 2

Diana (pseudonym) was trying to understand how the other students thought about the activity without any success:

> I still don't understand how they do it; because a couple of people tried to explain it to me but I was just completely lost, I didn't know how they could look at a number and go "alright that the 1 so the next one needs to be 2 [or] okay that, the next one needs to be 4". I still don't understand it, their way. (interview extract)

The teacher did not provide any guidance, which is something that frustrated her:

> And sometimes it's just really **frustrating,** because I think, well, "am I doing it right or am I doing it wrong?" and he says "what do you think?" And I'm thinking

> that "I don't know, that's why I'm asking you!" And it frustrates me. (interview extract)

I noticed her frustration while working with another group, I approached her, I saw her notes (e.g. Figure 3) and I realised that she was already engaged in a process of her own; thus I decided not to interrupt her and I chose to return to my previous group. The result was that she developed her particular understanding alone. Her first intuition, as it came out in her coursework was to "look at the rule" of the task:

> I understood the rule and what we were doing but for some reason I couldn't get my head around the flow diagram that everyone was doing [as in Figures 1, 2]. I started looking at the rule we had been given and what we had found from the other numbers we had used. I decided I would use a flow chart and above it write out the numbers from 1 to n-1 [see Figure 3]. This way I could see if there was more than 1 cycle and if there was more than 1 cycle I could cross the numbers out as I went along to make sure I had used them all. (interview extract)

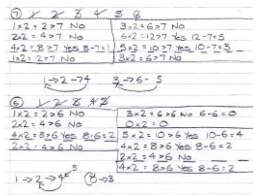

Figure 3

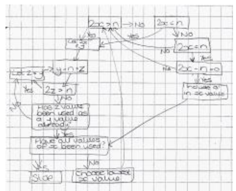

Figure 4

She illustrated the structure of the activity ("the rule") in her own way (Figure 3), as a trial-and-improvement process, in a question/answer (q/a) manner:

> …so I was having to devise this kind of system, where I've said "right, okay, if I do **this** to that, if I times this number by 2, is it bigger than the 9? No it isn't! So what do I have to do to it? Then I have to do this to it". (interview extract)

We notice her q/a approach in figure 3: Is 4>7? "No". Is 8>7? "Yes". Then, 8–7 = 1 (written on the same line as the previous q/a). Is 2>7? "No". When a 'loop' is completed the corresponding numbers are crossed at the top. This series of q/a written line after line for each number, gave a full account of what happens when one 'stumbles' on a number that has exceeded the fixed number, and it is a clear representation of the rawest possible mind process demanded by the core of the task (the 'rule'). She used the trial-and-improvement process with *immediacy, intentionality* and a *feeling of certainty,* and that is what makes it an *empirical* intuition according to Husserl, as it transformed her experience of her world-as-lived to successive questions and answers concerning numbers. Although the analysis could not go as far as finding an empirical equivalent of this intuition in her experience, we can say that its crucial feature is the *separation* of the q/a sequence from an experience or a collection of experiences, similar to a situation such as 'finding objects in the dark'.

She filled pages of questions/answers sequences for each natural number from 4 to 18. Finally, these structured questions/answers became a new layer of data for her, and she saw them as *commands* and *judgments,* similar to a computer algorithm. She constructed an algorithm attempting to describe the process for *any* given number (Figure 4). It was an intuition since, as Hintikka puts it, "according to Husserl such an abstraction of an essence can only take place in intuition" (Hintikka 2003, 181), and in Husserl's words, "[s]eeing an essence is therefore intuition" (ibid., 181). This time her intuition was based on the previous empirical one: the q/a material was a result of her previous empirical intuition; but the intuition that followed separated the algorithm form from the q/a material, and it was what Husserl describes as the 2nd intuitive stage, the intuition of essences *(Wesensschau).* For Husserl the intuitions of essences (abstract intuitions) are the ones that actually create abstract objects, and our example shows how abstract objects are formed on the basis of empirical ones, thus projecting an approach to knowledge that is essentially based on our lived experience, rather than internal mind processes. Hintikka's explication of Husserl's *intuitions of essences* finds here a clear instantiation since, "*Wesensschau*, too, pertains to objects, albeit objects different from sensible particulars. For Husserl "*Wesensschau* transforms sensory intuition into intuition of essences" (ibid. p. 182).

Implications for the student – Conclusion

The 'doubling modulo' activity and its treatment by the student had major implications for her, as the following extracts from her interview manifest:

> it was very very long-winded and it took me **a lot** longer to understand it than it did for everyone else. But now **I've done this** and **it's my way,** nobody else seems **to have done that way**. … I like my way! It's a long way, but I like my way … I think when I'm teaching **I need to get over the urge to give answers… not straight away.** … [A] lot of the time –and it's not just with maths, it's with other subjects– I've been very cautious when I found an answer, *if my answer isn't the same as everybody else's* I've just assumed **mine is wrong!** Because it's not the same as somebody else's … I think I've possibly become a bit more confident with what I'm doing; because *I'm thinking "well, because I've got a different answer to somebody else it doesn't necessarily mean that it's wrong, it just means I thought about it in a different way"*. So it has, yeah it's been good for me! … I think I have realised from these lessons that *it helps people more if you help with their understanding of it rather than just give them an answer straight away; and naturally make them think about it and what they are doing and why they are doing it,* which obviously is very important with wanting to be a teacher. (bold indicating her emphasis, italics added)

Starting from an empirical intuition and due to the character of the sessions that allowed a loose frame of what a 'correct' result is, the prospective teacher developed her ideas, devised a method, and cashed in the whole operation to a new perception of learning and teaching. She separated the question/answer sequences from her experience in her empirical intuition, which enabled the abstract intuition or intuition of essences (*Wesensschau*), where a new separation from the new layer of data took place, namely the separation of the algorithmic abstract essence from the particular patterns concerning 15 numbers (the hyle, or raw data in Husserl's terms), spread over several pages. Hintikka (1995, 183) explains:

> According to Husserl, we can come to know essences by means of empirical experience by separating them from the hyle in *Wesensschau.* The main difference as compared with Kant is that we do not create the essences, we merely come to know them by separating them from the sensory data. (Hintikka 1995, 183)

As the aforementioned extracts and further observation of the prospective teacher in this course as well as in other courses showed, she reconsidered her approach to learning and teaching in two major issues, as her "frustration" was radically transformed:

- Her results and methods can be valid although different than the other students' results and methods.
- Teaching and learning is not about giving or getting the answers "straight away", but rather about "help[ing] [people] with their understanding" and "making" the students "think about it and **what** they are doing and **why** they are doing it".

The research showed how new objects surface through empirical and abstract intuitions, and the interrelations between the two intuitive stages (empirical and abstract). Furthermore, we noticed how the materials of intuition are *neither internal nor external,* as there is an *overlap* of the objects as they are given in their **material form**—either drawn directly from experience, or as structured *questions and answers* concerning numbers—and their **intended form** that intuition separates from the sensory or abstract data (respectively). "[I]t is crucially important to emphasize that, according to Husserl, there is an actual interface of my consciousness and reality, that reality in fact impinges directly on my consciousness" (Hintikka 1995, 83) The materials of intuition are not prepackaged, but rather it is *reality impinging on consciousness* in the form of unstructured raw materials (Hintikka 1995, 102-103). This is how we have access to reality and not due to some predetermined internal mind structure. The mind-body dualism is cancelled, since consciousness for Husserl is embodied (Tito 1990, 186). And the relation between consciousness and reality is *mediated* by *intuition.* The phenomenological analysis may arrive at the empirical or abstract objects that *become known* by separating them from the unstructured raw material, rather than been *constructed* by discovering them in experience.

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