The use of Bloom's taxonomy in advanced mathematics questions

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Bloom's taxonomy for educational assessment has been regularly criticised by many in mathematics education for being particularly ill-fitting to mathematics, and yet continues to be used and discussed in this field. However, the Mathematical Assessment Task Hierarchy (MATH) was designed specifically for the development of advanced mathematics assessments in order to ensure that students are assessed on a variety of knowledge and skills. Here, I use MATH to contrast the types of questions posed in English A-level mathematics and further mathematics examinations with those in the University of Oxford's mathematics admissions test.

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The routineness of mathematics questions

At different levels, students will be asked questions of varying and increasing levels of complexity and difficulty. Pólya (1945) defined a problem as routine "if *it can be solved either by substituting special data into a formally solved general problem, or by following step by step, without any trace of originality, some well worn conspicuous example" (*171).

Routine questions have been found by Berry *et al.* (1999) to form the basis for the majority of marks awarded in mathematics examinations for A-level (the standard pre-university qualification in England) . Furthermore, when they redistributed the marks on pure mathematics papers to better reward solving non-routine problems, 297/311 of the scripts analysed had reduced marks. As such questions do not require original thought or application in new situations, it is perhaps understandable to find that such questions are those which lower-attaining students succeed with (Berry *et al.* 1999).

Conversely, non-routine questions require the application of mathematics in a new situation and/or creative thinking in finding a means to a solution. Studies have found that first year undergraduate students struggle with non-routine calculus questions despite possessing the knowledge necessary to be able to solve them (Selden, Selden and Mason 1994).

However, "routineness has to do with what the solver is used to" (Hughes *et al.* 2006, 91). Furthermore, one question may be considered routine in one instance, and yet non- in another, for example "a student who succeeds in proving an unseen theorem is demonstrating an ability to apply knowledge to new situations, but may only be demonstrating factual recall when proving it for a second time" (Smith *et al.* 1996, 68). This means that, whilst students could 'prove' theorems in examinations through their rote-learning in advance, they may also be capable of doing this in apparently non-routine questions had they seen similar questions posed in advance.

Bloom's Taxonomy

Perhaps the earliest taxonomy developed for educational assessment was that produced by Bloom *et al.* (1956). Designed for general application across all school subjects, many in mathematics education have deemed it particularly ill-fitting to mathematics (e.g. Kilpatrick 1993). More generally, it has also been suggested that Bloom's Taxonomy fails to identify levels of *learning* as opposed to designing different *types* of question (Freeman and Lewis 1998), and that its hierarchical nature is flawed, as certain levels in it may be considered interdependent (Anderson and Sosniak 1994).

Kadijević (2002) strongly encourages the operationalisation of taxonomies when designing assessment, which can be used to "guide and foster an adequate mathematics learning [and]… achieve a comprehensive evaluation of its outcome" (97). Various taxonomies have been proposed; some designed for general assessment, some for mathematics assessment. However, as with most educational frameworks education, caution must be exercised.

Little published empirical research in this area has statistically validated the use of taxonomies, meaning that its trustworthiness cannot be properly reflected in students' scores (Kadijević 2002). However, the most significant difficulty associated with using taxonomies relates to the classification process itself, specifically:

- **It is difficult to put certain questions into just one category.** More involved questions can include routine and procedural calculations as part of the solution process.
- **It is difficult to know what skills and thinking are employed by individual students to answer a question.** For example, when asked to prove a theorem, a student may:
  1. learn the proof by rote and reproduce it from memory when assessed; or
  2. understand the principles and associated concepts and definitions, and use these to independently develop a proof.

Mathematical Assessment Task Hierarchy

One of the more relevant modifications of Bloom's Taxonomy to undergraduate mathematics study was conducted by Smith *et al.* (1996), where their focus was on the *skills* required to complete a particular mathematical task. Their Mathematical Assessment Task Hierarchy (MATH) (Smith *et al.* 1996) was designed to assist the development and construction of advanced mathematics assessments in order to ensure that students are assessed on a variety of knowledge and skills.

Categories

The MATH categories are designed in order to describe the "*nature* of the activity… not the degree of difficulty" (Smith *et al.* 1996, 68). That is, a Group A task may be considered more difficult than a Group C task by a particular student, depending on their perception of difficulty, as well as the particular challenges associated with the task.

Group A	*Routine procedures*	
	Factual Knowledge and Fact Systems	Recall previously learnt information in the form that it was given.
	Comprehension	Decide whether conditions of a simple definition are satisfied, understand the significance of symbols in a formula and substitute into that, recognise examples and counterexamples.
	Routine Use of Procedures	Use a procedure/algorithm in a familiar context. When performed properly, everyone solves the problem correctly, in the same way. Students were previously exposed to these in drill exercises.
Group B	*Using existing mathematical knowledge in new ways*	
	Information Transfer	Transfer information from verbal to numerical or vice versa, decide whether conditions of a conceptual definition are satisfied, recognise applicability of a generic formula in particular contexts, summarise in non-technical terms, frame a mathematical argument from a verbal outline, explain relationships between component parts, explain processes, reassemble given components of an argument in their logical order.
	Application in New Situations	Choose and apply suitable methods/information in new situations.
Group C	*Application of conceptual knowledge to construct mathematical arguments*	
	Justifying and Interpreting	Prove a theorem in order to justify a result/method/model, find errors in reasoning, recognise limitations in a model, ascertain appropriateness of a model, discuss significance of given examples, recognise unstated assumptions.
	Implications, Conjectures and Comparisons	Given or having found a result/situation, draw implications and make conjectures, and justify/prove these. The student can make comparisons, with justification, in various mathematical contexts.
	Evaluation	Judge the value of material for a given purpose based on defined criteria which may be provided or need to be determined.

Table 1 – Categories in the MATH (adapted from Smith *et al.* 1996)

Uses

The mathematical skills associated with Group C – "those that we associate with a practicing mathematician and problem solver" (Pountney, Leinbach and Etchells 2002, 15) – are those which, unfortunately, have been found to be most lacking amongst undergraduate mathematicians (Ball et al. 1998; Smith et al. 1996). Similarly, Etchells and Monaghan (1994; cited by Pountney *et al.* 2002) found that A-level mathematics examinations awarded marks mainly for Group A tasks.

Some have questioned whether assessment which passes students who only have Group A and B skills should be permitted, proposing that high marks should only be available to those with Group C skills (Pountney, Leinbach and Etchells 2002). Leinch, Pountney and Etchells (2002) suggest that Group B and C skills should be gradually introduced and utilised in such a fashion that "they become *for that individual* student Group A… because they have developed the insight into the problem-solving process that makes the solution of the problem straightforward" (13).

Group	Category	A-level Example	Undergraduate Example
Group A	FKFS	State the cosine rule.	Let $f: X \to Y$ be a function and $B \subseteq Y$. Define what is meant by $f^{-1}(B)$, the inverse image of B under f.
	COMP	Given that $zz^* = 9$, describe the locus of z.	If the function f is continuous on the interval (a, b) but not bounded then $\int_a^b f(x)\, dx$ does not make sense as a proper Riemann integral. Briefly explain why not.
	RUOP	A curve's equation is $y = f(x)$, where $f(x) = \frac{3x+1}{(x+2)(x-3)}$. Express this in partial fractions.	Use L'Hôpital's Rule to find the limit of $\left(\frac{\log(n^3+1)}{\log(2n^5-1)}\right)_{n\in\mathbb{N}}$
Group B	IT	Describe a sequence of two geometrical transformations which map $y = x^2$ onto the graph of $y = 50 - x^2$.	Describe, in about 10 lines, the ideas of the Mean Value Theorem. Imagine that you are describing the theorem to a student about to start university.
	AINS	Using a sketch, show that $y > \tanh(\frac{y}{2})$ for $y > 0$ and deduce that $\operatorname{arcosh} x > \frac{x-1}{\sqrt{x^2-1}}$ for $x > 1$.	Prove that $\max(a, b) = \frac{1}{2}(a+b) + \frac{1}{2}\lvert a-b\rvert$
Group C	J&I	The matrix A is $A = \begin{pmatrix} 3 & 1 \\ 0 & 1 \end{pmatrix}$. Prove by induction that, for $n \geq 1$, $A^n = \begin{pmatrix} 3^n & \frac{1}{2}(3^n - 1) \\ 0 & 1 \end{pmatrix}$.	Explain why the Mean Value Theorem does not apply to the function $f(x) = \lvert x+1\rvert$ on the interval $[-3, 1]$.
	ICC	The fact that $6 \times 7 = 42$ is a counter-example of: • The product of any 2 integers is odd; • If the product of 2 integers is not a multiple of 4 then the integers are not consecutive; • If the product of 2 integers is a multiple of 4 then the integers are not consecutive; or • Any even integer can be written as the product of 2 even integers.	Prove that if f and g are continuous at x_0, then $\max\{f, g\}$ is continuous at x_0. What about $\min\{f, g\}$?
	EVAL	Given a particular function, discuss the accuracy of the trapezium rule in finding the area under the curve.	The Mean Value Theorem is a powerful tool in calculus. List 3 consequences of the Mean Value Theorem and show how the theorem is used in the proofs of these consequences.

Table 2 – Example questions classified using the MATH (sources for these examples on request)

Comparing A-level mathematics and the University of Oxford Admissions Test

C1 and FP3 papers were analysed in order to provide a comparison across papers from mathematics and further mathematics A-level and a variety of examination boards. In common with existing research, the majority of the questions were found to be Group A, most commonly a 'routine use of procedures'. There were very few Group C questions in the papers analysed, with most of them proofs by induction in FP3. Whilst such questions do fit with the outline of 'implications, conjectures and comparisons', A-level proof by induction *could* be reduced to a routine procedure, making this question Group A. For the purposes of this research, 'contentious' questions were classified according to a 'worst case scenario' wherein questions would be labelled Group A over Group B/C and Group B over Group C.

The Oxford Admissions Test (OxMAT) is sat by candidates of either single or joint honours mathematics to the University of Oxford. There is no standardised pass rate, but it is used in to inform admissions tutors' decisions. The first question is always a ten-part multiple choice question, with the remaining questions being much longer, requiring candidates to show their working. Questions require, at the most, knowledge to the level of the C1 and C2 A-level mathematics.

In order to identify the nature of the questions that the test poses to candidates, the MATH was applied to the last five years' papers which are available online. This analysis found that the majority of OxMAT questions are from Group C, and the minority Group A. Group A questions comprised of those which were small parts of larger questions. However, these formed a minority of marks. Multiple choice questions covered all classification groups.

RUOP (Group A)	AINS (Group B)	ICC (Group C)
The smallest value of $$I(a)=\int_0^1 (x^2-a)^2\,dx$$ as a varies, is (a) $\frac{3}{20}$ (b) $\frac{4}{45}$ (c) $\frac{7}{13}$ (d) 1	The point on the circle $$x^2+y^2+6x+8y=75,$$ which is closest to the origin, is at what distance from the origin? (a) 3 (b) 4 (c) 5 (d) 10	In the range $0 \leq x \leq 2\pi$, the equation $$2^{\sin^2 x}+2^{\cos^2 x}=2$$ (a) has 0 solutions; (b) has 1 solution; (c) has 2 solutions; (d) holds for all values of x.

Table 3 – 2009 multiple choice questions classified using the MATH

In interviews conducted as part of a larger study, current undergraduate mathematicians described how they believed the OxMAT to test a very different skill to A-level mathematics. Whilst quantitative analysis was not conducted on the findings from the application of the taxonomy to the A-level or OxMAT questions, it is clear that there are substantial differences between the two. The mathematics required to answer the OxMAT questions correctly is not more advanced than that at A-level, but instead requires its application in unfamiliar circumstances. The mixture of questions at A-level is very limited, with the vast majority being from Group A and small numbers from Groups B and C. In fact, one C1 paper had 67 of its 75 available marks awarded for answers to Group A questions. Conversely, there is a wide variety of questions from the different groups in the OxMATs, affording students the opportunity to demonstrate the breadth and depth their mathematical skills.

Reflections and conclusion

This initial analysis revealed there to be a majority of questions at A-level which fit into Group A, whereas the OxMAT's questions fall into a mixture of A, B and C, with the majority being from Group C. This discrepancy in the types of questions asked appears to be great and is worthy of further investigation. I am currently in the process of analysing undergraduate mathematics examinations using the MATH in order to provide further means of comparing the types of questions that students are posed at different levels of their education, with a view of ascertaining whether this shapes their experience of the subject and perceptions of what it is to learn mathematics.

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