Making sense of fractions in different contexts

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This presentation is based on a study of 20 pupils, aged 9-10, in a Norwegian primary school. The pupils were exposed to two, rather different, classroom situations and in both situations the concept of fraction was central. The pupils were given tasks and in order to accomplish these tasks it was necessary to make sense of fractions in some way. An interesting observation is how the presence of different mediating artefacts influences the pupils' meaning making.

Keywords: Fractions, semiotic representations, mediating artefacts.

The classroom episodes

The first episode takes place in the pupils' regular classroom, which is quite large and holds an area furnished as a kitchen at one end. There are 20 pupils in the class and the pupils come in groups of five to the kitchen area to do a particular task, making batter for waffles. This task involves measuring out a number of ingredients (milk, flour, butter) and in this paper I am particularly interested in what happens when the pupils measure out 15 decilitres of milk. The milk comes in boxes marked "1/4 liter[2]", and the pupils have measuring beakers available that can take 1 litre. The beakers are transparent, with a scale marked "1 dl, 2 dl, …. 9 dl, 1 lit" from bottom to top.

The second episode takes place some time later. In this episode the pupils receive a task sheet with drawings of red and blue milk boxes of equal size and with the information that a blue box contains 1/3 litre of milk and a red box contains 1/4 litre. Here the standard fractional notation with a horizontal bar is used. In this text I will use the fraction notation *a*/*b* to save space. On the task sheet the following four situations are described: A: Three blue boxes, B: Four blue boxes, C: Four red boxes, and D: Three red boxes. The following questions are given:

- Which box, red or blue, contains most milk?
- Which situation, A, B, C or D, represents the largest quantity of milk?
- And which situation represents the smallest quantity of milk?
- Are there any situations with the same amount of milk?
- How many decilitres of milk are there in situation D?
- You need 15 decilitres of milk and you have boxes containing 1/4 litre, hence red boxes. How many boxes do you need?

The pupils have only pencil and paper available and no concrete material. The 20 pupils are grouped in the same way as in the first episode and each group leaves the rest of the class to join me in an adjacent room to work with the task for about 30 minutes.

Both episodes were video recorded and the video footage constitutes the most important data for the analysis. Video recordings have been transcribed, first in Norwegian, and later parts of the transcriptions have been translated into English. In addition there exist notes and drawings made by the pupils in the second episode.

[2] Norwegian spelling of litre

The most important research questions for the study are: *how do the children make sense of fractions given with different representations*, and, *in what ways will mediating artefacts influence the children's sense making of fractions?*

Theoretical framework

The study reported on in this paper is concerned with pupils applying and developing mathematical knowledge in different settings, which calls for a stance that knowledge is situated (Lave and Wenger 1991). Closely related to this is also the idea that the knowledge depends on the sociocultural artefacts that mediate between stimulus and response (Wertsch 1991). I will use the term artefacts to denote both physical tools, such as measuring devices that are used in the described situations, and psychological tools, such as language and signs. All the artefacts involved are considered as cultural tools, containing both psychological and physical aspects (Säljö 2005/2006, 28).

My analysis of the pupils' work in the two situations rests heavily on semiotic theory. The concept of *sign* is fundamental, and according to Peirce

> [a] *sign* is a thing which serves to convey knowledge of some other thing, which it is said to *stand for* or *represent.* This thing is called the *object* of the sign; the idea in the mind that the sign excites, which is a mental sign of the same object, is called the *interpretant* of the sign. (Peirce 1998, 13, emphasis in original)

A sign has two functions, a semiotic function; "something that stands for something else", and an epistemological function, indicating "possibilities with which the signs are endowed as means of knowing the objects of knowledge" (Steinbring 2006, 134). All mathematical objects are abstract but, despite this, mathematical concepts and the signs representing them are used to refer also to real life situations. A sign or symbol can therefore be thought of as representing a mathematical concept as well as a concrete object or reference context. This is visualised in *The Epistemological Triangle* (Steinbring 2006, 135) shown in Figure 1 below.

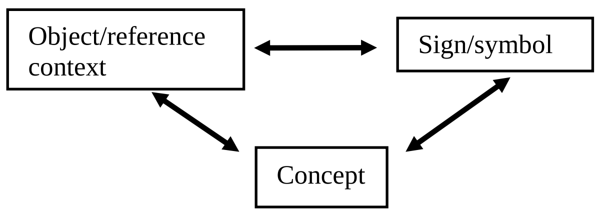

Figure 1: The epistemological triangle

The relations in the Epistemological Triangle are largely conventional and in a learning process these relations are in development. In a given situation meaning is created through mediation between the sign/symbol and object/reference context. This means that the system is continuously in development based on interaction between pupil/s and teacher. According to Steinbring "[t]he links between the corners of the epistemological triangle are not defined explicitly and invariably, they rather form a mutually supported, balanced system" (Steinbring 1997, 52).

Analysis of the two episodes

Although the two episodes can be said to deal with largely the same mathematical topic they are very different in their context. Even though the first episode takes place in a mathematics lesson it is very close to an everyday context. Both the task itself and the artefacts that are used are of a nature that the children will recognise from their daily life experiences. The purpose of the task is also of a very practical character. The pupils were supposed to make the batter for the waffles and later in the day they were going to cook the waffles and eat them together with other pupils at the school. In the second situation the task is of a nature which could be said to be typical for a school task. Although it makes reference to daily life artefacts (milk boxes) the milk boxes are only imaginary and it had no practical consequences whether the task was correctly solved or not.

Making sense of the symbol 1/4 liter

In the first episode most pupils noticed the text "1/4 liter" on the milk boxes and they started to discuss the meaning of this sign. Several suggestions were offered for the meaning of the sign. Here are some examples: "One four litres", "One comma[3] four litres", "four and a half litres", "one and a half litres". Some of the suggestions are combined with common sense such as when the teacher challenges the proposal that it is 4.5 litres in one box the pupils suggest that it must be decilitres, because, as one pupil says, "it isn't even half a litre". In one of the groups Jessica suggests that one box contains "one comma four" (i.e. 1.4) litres and James follows up by suggesting that it will be 2.8 if they take two boxes. If they had relied on counting in this way they would not have obtained the desired amount of milk but Ellie points to the fact that there is an empty measuring beaker on the table which they can use. Jessica had not seen this in the first place, but being made aware of it she and James start pouring milk into the measuring beaker. Now the scale of the measuring beaker takes over the role as a sign connecting the amount of milk to the boxes (reference context). This new sign renders the original sign 1/4 liter obsolete and the pupils no longer have any need to make sense of this sign. On the video one can see that the pupils follow closely the level of milk rising in the measuring beaker when they pour in the fourth box and they show no sign of making a connection between the fact that they have used four boxes and that the scale shows 1 litre. Jessica says that "we need to have five more decilitres". Now they pour the milk into the bowl with the flour and fetch another box of milk. Jessica pours in the content of the box into the now empty measuring beaker and says "three decilitres" while looking and pointing at the scale. During this process I ask how much there is in one box. Jessica looks at the sign 1/4 liter and says "one comma four litres".

The measuring task has now been completed without ever making correct sense of the sign 1/4 liter. When I ask them how many boxes they have used they present the answer "six" which is obtained by counting the empty boxes. The excerpt below shows how Jessica works only in the realm of decilitres and the number of boxes only comes in because it is explicitly asked for by me, not because it is necessary to complete the task.

Jessica:	Three, four, five six. Have you thrown away any?
Ellie:	Me. No.
Jessica:	OK, then we have used six.
Frode:	Six, to get 15 decilitres?
Jessica:	Yes, we had 10 before, and then we took five now.

[3] In Norwegian "comma" is used for the decimal point, so "one comma four" would be 1.4.

My interpretation of this episode is that during the process the measuring beaker has been introduced as a new sign, replacing the sign given in fractional notation, and the mediation of the concept 15 dl takes place between the scale of the measuring beaker and the milk boxes instead of between the sign 1/4 liter and the milk boxes. The effect of the measuring beaker can be seen also in the reasoning of Joseph and Thomas, who were urged by the teacher not to use the measuring beaker.

Joseph:	Ohh. A quarter of a litre, that is … a quarter … ten decilitres is one litre. We have to have three of these then, then it will be. Five of these I think … no not five. How much should we, Thomas, if we take three of these, no four, then it is one litre and we want fifteen decilitres, and that is, and ten decilitres that is one litre. But how many more than four do we have to take then?
Thomas:	Then we have to take four, and then we have to take … two
Joseph:	Then we have two, and ten decilitres here. And then it is fifteen.

The excerpt above shows that 1/4 is replaced by “a quarter” and that “four plus two boxes” will equal one and a half litre. Without the measuring beaker the sign 1/4 is given meaning in order to solve the task.

Which box, red or blue, contains most milk?

This is the first question on the sheet given to the pupils in the second episode. Here the reference context is taken to be the pictures of the coloured, equally sized, boxes and the sign is given in the standard fraction notation, such as in Figure 2.

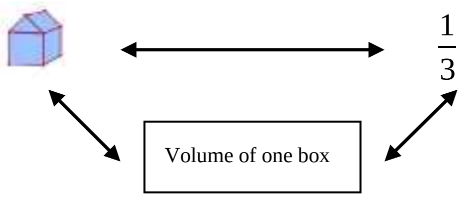

Figure 2: The epistemological triangle for the volume

To compare the volume of the red box to the volume of the blue box the only available representation is the symbol given as a fraction. The reference context gives no information about the relative size of the boxes. The pupils soon agree that 1/3 > 1/4 and to justify their argument they create a new reference context in terms of a rectangle divided in strips. An example is shown in Figure 3. The drawings are not made to match the actual situation but I interpret that the drawings are meant to show that when $m > n$, $1/m < 1/n$. This interpretation is supported by a statement from one of the pupils saying, “the larger the number below, the smaller is the actual part”.

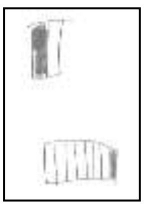

Figure 3: Comparing fractions

How many boxes to get 15 dl?

In Episode 1 the pupils relied on the measuring beaker to get the correct amount of milk, except Joseph and Thomas who were encouraged to manage without it. In Episode 2 the option of using a measuring beaker, or any other physical artefact, is not there, so the pupils have to rely on making sense of the signs. I have previously described how the group with Jessica and James completely depend on the scale of the measuring beaker to get 15 dl in Episode 1 and that they answer the question about how many boxes they have used by just counting the empty boxes. Below is part of the dialogue when the same group solves the problem of getting 15 dl in Episode 2.

Jessica: Five, ten, fifteen. It will be three

Ellie: So it is three

Frode: OK, five, ten, fifteen. That is three

Ellie: It is just like in D, one, two, three

Jessica: Three boxes, it will be three boxes

James: Three boxes, no, we should have fifteen

Ellie: We are not supposed to have fifteen boxes, but fifteen decilitres

Jessica: Yes, and each box takes two and a half decilitres

Emily: Couldn't we…

Jessica: I did not understand this

Ellie: Me neither

Emily: Two comma five, two comma five, that is five, and then we have five three times in fifteen, and then it is two for each, so it is six

Ellie: OK, but then I did not understand anything

…

Emily: Every five is two boxes, so it is three, therefore six.

After some initial confusion Emily comes up with a solution by converting 2 times 2.5 to 5 and then 3 times 5 to 15. Then she finds the number of boxes, 6, by taking 2 times 3. Compared to the solution by Joseph and Thomas presented before there are similarities but also differences. Both solutions entail building up the total amount using a multiplicative procedure but in different ways. Joseph and Thomas find that 4 boxes equal 1 litre and that they need 2 more to get 15 dl = 1.5 litre. This reasoning was repeated by Joseph in Episode 2 when he said about Situation C (where there are 4 boxes of 1/4 l): "C is one litre, which is ten decilitres. Then I need half of C again, and that is two and therefore it is six. Two plus four is six." Both solutions involve two steps, where the first step establishes a relation between a number of boxes and a number of decilitres that is easy to handle further to get 15. Presented in a table the two solutions can be illustrated as shown in Tables 1 and 2 below.

Boxes	Dl
4	10
4+(half of)4= 4+2=6	15=10+(half of)10= 10+5

Table 1: Joseph's solution

Boxes	dl
2	5
3x2=6	15=3x5

Table 2: Emily's solution

The solutions shown in Tables 1 and 2 end with the answer *six boxes* as a result of a process where the symbol “1/4 liter” plays a crucial role. Joseph, in accordance with his reasoning in Episode 1, connects 1/4 with “a quarter” and “four quarters equal one whole (litre)”. Emily links decimal notation to fractional notation, 2.5 dl = 1/4 l, and then she uses 5 dl as a starting point to count 5-10-15. In Episode 1 the presence of the measuring beaker made the interpretation of the symbol 1/4 liter redundant. Instead of 1/4 liter being the sign that mediates between boxes and decilitres the scale of the measuring beaker was used as the mediating artefact. The scale functions as an *indexical sign* (Peirce 1998) that has a real connection to the object that it represents, namely the milk in the measuring beaker.

Further analysis of the two situations described in this paper can be found in Rønning (2010 and in press).

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