Exchange as a (the?) core idea in school mathematics

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I propose that *exchange* is a core idea underlying much of school mathematics. Alerted by young children struggling with the difference between coins as objects and coins as having value, I began to explore the action of exchanging one thing for another. If *exchange* is augmented to include *substitution* then it shows up everywhere, from counting to algebra, from money to currency, from ratio to algorithms and Turing machines.

Introduction

The phenomenon of interest is children in years 2, 3 and 4 who when shown some play-coins and asked "how much is there?" respond by counting the number of objects rather than adding their total value. Primary teachers have been quick to tell me that young children do not get to use coins in the way they themselves did when young, because of credit cards etc.. Nevertheless there is an important awareness which underpins not only mathematics but ordinary life, in which things have value(s) and sometimes you are expected to attend to the quantity and sometimes to the value.

I began the session therefore with the observation that prior to the act of counting, which requires coordinating the physical action of pointing with the verbal act of reciting a memorised cultural poem, there is the physical action of exchanging one thing for another, repeatedly. Thus

Task 0: I have a pile of red counters (all the same size) and you have yellow counters. Exchange each of my red counters for a yellow counter until all the reds are gone. What mathematical action is involved?

At the heart of this action is the awareness of one-to-one relationship. Here I am using *awareness* in the sense of Gattegno (1987; see also Young and Messum 2011) to mean 'that which enables action'. However, the action of exchange depends on discerning and distinguishing both the entity-ness of individual counters, the colours of the counters, and distinguishing the red counters from each other without being concerned about minor imperfections in the colouring or the shape. It also requires some fine motor control, and sufficiently focused attention to complete all the exchanges, repeating the exchange action over and over. Finally, there is an expectation that repetition of the act of exchange is not simply a repeated physical act, but is accompanied by some sort of growing sense of the act of exchanging 'one thing for another'.

In this and the following tasks my question is about the mathematical action, but this question is for the teacher not the child!

Tasks involving such an exchange can be set in many different contexts, changing the red counters to other objects. Also there can be a practice of lining up the reds and the yellows as the exchange takes place. Someone commented that the language of this task might be demanding for young children; however here I am concerned with the mathematical awarenesses. I leave to primary experts how to phrase such tasks. I am confident that children will quickly learn what exchange means through being immersed in such tasks.

At some point these exchanges become related to the act of counting (uttering items from the verbal 'poem' consisting of number names) so that cardinality becomes available as a focus of attention in exchange tasks.

Task 1: I have a pile of red counters. I exchange each of them for 3 yellow counters. What mathematical action is involved?

The underlying awareness is what we (later) call multiplication. As Dave Hewitt observed, if you attend to the exchange, you experience scaling (one to three); if you attend to the growing pile of yellows you experience repeated addition. These are two vital aspects of multiplication, but scaling gets overlooked when children are led to believe that 'multiplication is repeated addition' rather than that 'repeated addition is one form of multiplication'. Note that engaging in one or two similar exchanges is preliminary to but not the same as internalising a deep sense of exchange, and different again from becoming consciously aware of the generality that is being instantiated: any number of red counters, exchanging them for some specified number of yellow counters. So far so good.

Task 2: I have a pile of red counters. I exchange 5 reds for 1 yellow and do this until I can make no more exchanges. What mathematical action is involved?

With some encouragement people responded with terms such as ratio, division and division with remainder. At some point in this sequence one could invite children to exchange, say 5 small red counters for 1 large red counter. The notion of 'value' arises from context (a large red is 'worth' 5 small counters) as a subsidiary but important awareness. Note however that the relative sizes of coins do not indicate their relative value. Thus it is vital when attending to size to vary whether the larger counter is worth more or less. This can be augmented by having large objects worth the same or less than smaller objects when engaging in play-shops and other exchange activities.

The task can be augmented by inviting children to explore what numbers of red counters, once exchanged, end up with only yellow counters, or with exactly 1 yellow counter.

Now things get a bit tricky.

Task 3: I have a pile of red counters. I exchange 5 reds for 2 yellows and do this until I can do no more exchanges. What mathematical action is involved?

Different ways of attending to the action might lead to different awarenesses. For example, there is a doubling and a dividing by 5. If there is a remainder then the left over reds are 'worth' 2/5ths of a yellow, so perhaps what is going on is multiplication by 2/5, or multiplying by 2 and dividing by 5. However:

Task 4: I have a pile of red counters. I exchange 1 red for 2 browns, and 5 browns for 1 yellow. What mathematical action is involved?

I have a pile of red counters. I exchange 5 reds for 1 green, and 1 green for 2 yellows. What mathematical action is involved?

Essentially, the result can sometimes depend on order: if you double first and then divide by 5 you may have some brown left over; if you divide by 5 first you may have some red counters left that cannot be exchanged. For example, starting with 18 reds, the first exchange rules end up with 7 yellows and 1 brown, while the second exchange ends up with 6 yellows and 3 reds. How are these to be reconciled?

It depends on having absorbed the notion and language of value, so that seeing each red as worth 2 browns, 3 reds are worth 6 browns which is 'the same as' 1 extra yellow and 1 red remaining. Thus the exchanges can be reconciled, but, I suspect, only after having encountered and developed some fluency with the language of 'value'. This in turn can be supported by being immersed in many simpler exchanges in many different contexts over a considerable period of time. Helen Williams (workshop at ATM Easter 2012) has videotape of children engaged in a variety of exchange tasks in different contexts, ending up with an auction in which it seems that at least some of the children haven't really grasped what bidding is about!

It is worth noting that Valerie Walkerdine (1988) challenged the practice of using unrealistic values for pretend objects when trying to get children to work with tasks.

I then went on to provide evidence that exchange, often in the form of substitution, pervades school mathematics. A slight difference between these notions is that for some people exchanges are reversible, while substitutions may not be.

Barter and Exchange

Barter has taken place long before and well after the introduction of money. For example, there are amazingly complete records of exchanges in the town of Prato (now a part of Florence) over two hundred years (Marshall 1999, 72-73). Here are three instances:

Task 5 : I will exchange 3 of my sheep for 5 of your geese; I know I can exchange 7 geese for a colt …

As a baker I will exchange 12 loaves of bread for use of your horse for a day

As more and more people became merchants, it was necessary to educate sons into the mechanics of barter. The renaissance painter Piero de la Francesca (1412-1492) was asked by his patron to write a book for young men to learn arithmetic and in it there are tasks such as

Task 6: Two men want to barter. One has cloth, the other wool. The piece of cloth is worth 15 ducats. He puts it up for barter at 20 and 1/3 in ready money. A cento of wool is worth 7 ducats. What price for barter so that neither is cheated?

I had to be helped to see that what it means is that the barter price is 20 but that 1/3 must be in cash (this at a time when coins were scarce). The solution provided involves dividing 56 by 5:

> Treat the 1/3 ready money as 1/3 of 20, that is 20/3. Reduce both the original and the barter prices by the amount of ready money: 15 – 20/3 and 20 – 20/3, namely 25/3 and 40/3. The ratio of these gives the 'inflation' proportion required, namely 40/25 = 8/5 (and involves a division by 25/3). Then the wool merchant should barter at 7 x 8/5 = 56/5 ducats, and this agrees with the answer given by Francesca.

Adolescents sometimes like collecting 'cards' showing football players or the like, and they engage in swaps such as "You can have any three from this pile in exchange for … (two specific cards)".

Functions

Whenever there is an input-output relationship there is a form of exchange, or at least substitution going on: I exchange an x for the value associated with x. This applies (without any formal language using x or f) to look-up tables such as timetables, vending machines and so on. Often the exchange is one way: it cannot be reversed, or cannot be reversed uniquely.

It is well known that the awareness which underpins functions and the language of $f(x)$ for functions remains mysterious for many students. Graphs of functions are literally the coordination of input with output, which can be 'seen' as a form of exchange.

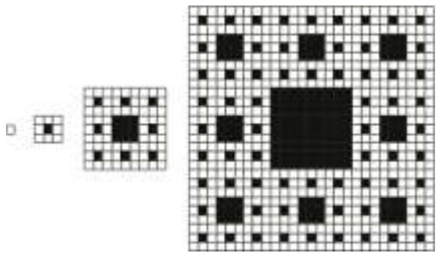
Attention directed to functions, as elsewhere in algebra, often focuses on the mechanics of manipulating symbols. Thus substitution into functions to find the function values presents obstacles to students who have no mental images with which to make sense of the act of substitution; perhaps exchange could provide that enactive foundation. A plausible conjecture might be that with extensive experience of exchange, and having integrated the discourse of exchange into their vocabulary, students might not find the notion of function so abstract.

Task 7: If a configuration of n identical hexagons forms a shape with 4n + 2 edges on its perimeter, how many edges will be made by 3n + 2 such hexagons in the corresponding shape?

The multiple use of n is an obstacle for many, when all that is signalled is exchanging each n in the formula $4n + 2$ with the expression $3n + 2$. Imagine the tension for Scandinavian countries in which the pronunciation of the words for 'one' and the letter 'n' are very hard to distinguish!

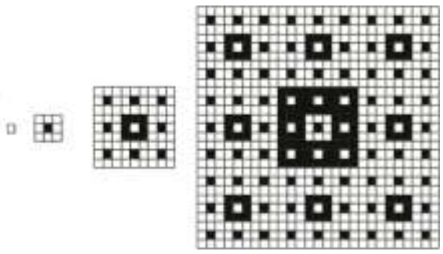
Patterns and Relationships

Task 8: You are shown the first three terms of a sequence of black and white pictures, each generated from the previous by means of the same rule. How many little squares will there be in the nth picture and what will be the proportion of black squares?



What might be interesting in this task is to catch yourself looking for and trying to articulate a relationship, which is presumably what people mean by 'pattern spotting'. If the relationship is 'the same' between each successive pair, then there is a property which is being instantiated, and that will serve to generate pictures farther along in the sequence.

In the session there was little time so I directed attention to the way in which each picture after the first is used to generate the next picture. Then I showed the second sequence so that participants could rehearse that particular way of looking.



The underlying perception is that each square is replaced with a 3 by 3 square, coloured according to a specific and invariant rule. A great deal of 'pattern spotting' that is currently used to stimulate pre-algebraic thinking involves substitution of something for something else, and expressing that relationship as a property.

Newton's Principle

Newton formulated an awareness that people develop spontaneously through enaction, even if they do not articulate it the way Newton did: if you have a collection of masses, then you can treat the system as a single mass (the sum of the masses) concentrated at the centre of mass of the system. Statics as a part of mechanics depends on this observation. But there are some slightly counter-intuitive aspects!

Task 9: Where is the centre of mass of three equal masses placed at the vertices of a triangle, or four at the vertices of a quadrilateral?

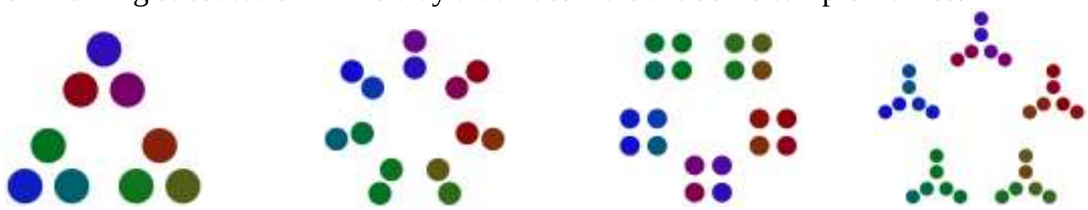
Where is the centre of mass of three rods forming the edges of a triangle, or four rods being the edges of a quadrilateral?

Where is the centre of mass of a uniform sheet of triangular (quadrilateral) material?

It turns out that for a triangle two of these must coincide but the third only coincides for special triangles, while for quadrilaterals, all three are in general slightly different. (Thanks to an ATM workshop led by Jayne Stansfield for reminding me of this.)

Number Necklaces

I tried to show an animation from the internet (Von Worley 2012) which displays *n* circles distributed around a circle in such a way as to display all the factors of *n*. My question was going to be "what can one do with this?" and whether participants saw it as involving substitution in the way that I do. Here are some sample frames:



Frames for 9, 14, 20 and 30

Actions

Whenever a mathematical investigation proceeds by locating and working with actions that preserve some property in the objects acted upon, there is a form of exchange going on. Any configuration can be replaced by the result of one of the actions. Mathematical attention then focuses on the actions and how they are related. For example, the inverse relation between addition and subtraction is a relation between the actions of 'adding *n'* (for some *n*) and 'subtracting *n*', and likewise for multiplication and division, exponentials and logarithms, differentiation and integration. Furthermore, the properties of arithmetic (commutativity, associativity, distributivity) that provide the properties for manipulating algebra, are relationships between actions, and can again be seen as a form of exchange.

Statistics

Newton’s principle alerted me to the fact that every ‘statistic’ is a summary of a set of data, and as such it stands for or (re)presents the original data. We exchange the mass of data for statistical information such as the mean, median or mode, but also upon occasion the maximum and the minimum.

The Whole of Algorithmic Mathematics

My final example involves Markov Sequences related to Post Productions. For example

Task 10: Given a sequence of symbols such as $AAAAAAAA$BBBBB$, you are permitted to replace any occurrence of A$B by AA$ and any occurrence of $$ by $. What mathematical action is being enacted by carrying out all possible replacements, over and over?

Interpreting $AAAAAAAA$ as a presentation of 8, and similarly for the Bs, the replacement rule effectively calculates the sum of two numbers. Now construct a similar replacement rule that will subtract two numbers. A little thought coupled with appreciation of the previous example leads to the rules A$B is to be replaced by $, and $$ is to be replaced by $. Finding a way to multiply and divide is rather trickier but can be done. Furthermore, the action of any Turing machine can be presented by replacement rules like these, so that exchange lies at the heart of all algorithmic mathematics.

Summary

As with all mathematical topics, what matters is not the specific exercises or tasks, but provoking students to be aware of the generality being instantiated.

Exchange certainly lies at the heart of the awarenesses that underpin counting and basic arithmetic. It seems that in the form of substitution it underpins much of school mathematics. The examples of exchange presented here were meant to illustrate the pervasiveness of exchange in school mathematics, and are certainly not exhaustive. Might it be the case that real appreciation of and familiarity with exchange in the early years could provide the foundation for many more students to find mathematical thinking both attractive and understandable? Yet to be considered is whether geometrical thinking involves exchange in any substantive way.

References

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