The role of justification in small group discussions on patterning.

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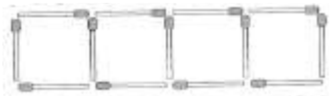
Swedish students have not been successful in solving geometrical pattern tasks in the TIMSS study and as a result it has been introduced as explicit core content in the National Syllabus (Lgr11) for grades 1-6. Analysis of video recordings of three student groups working with a task taken from TIMSS07 showed that students' initial approach to the task was often unsuccessful. In this situation it was then a call for justification that led them on, for example through questioning why a solution was correct or what the answer meant. The call for justification came from the teacher, from other students or from a student's wish to understand. An implication of this study is that students would benefit from incorporating justification as an essential part of their problem solving process.

Key words: algebra, patterns, justification, video data, TIMSS tasks, problem solving

Introduction

Generalizations and patterns are often highlighted as key ideas in mathematics, essential parts of early algebra and fundamental to algebraic reasoning (e.g. Cai and Knuth 2011). Working with problems of detecting and/or generating patterns, describing a term and its position in a sequence, is an approach to *algebra as generalization* aimed at enhancing students' insight into detecting sameness and differences, making distinctions, repeating, ordering, classifying and labeling (Mason 1996, 83). Lee (1996, 106) writes: "As an introduction to algebra, an entry into the culture, I think a generalizing approach is grounded historically, philosophically, and psychologically and has proven its merits pedagogically wherever it has been tried."

Although patterning has been acknowledged in school curricula in many countries it has not been a prominent part of Swedish school textbooks or of classroom practices. As a result of declining results on TIMSS tests, particularly concerning algebra, patterning was explicitly introduced as core content for grades 1-6 in the Swedish National Curriculum *Lgr11* (Skolverket 2011). In the rather short mathematics syllabus part of the curriculum (pages 59-63 cover the whole syllabus for mathematics in grades 1 through 9) patterning is mentioned twice, as core content for grades 1-3 as well as grades 4-6, in the short, and quite general phrase: "How simple patterns in number sequences and simple geometrical forms can be constructed, described and expressed." Currently patterning problems are making their way into textbooks, and teachers are starting to do patterning in their classes. A general question to ask in this situation is if teachers understand what students are supposed to learn by doing patterning tasks. Will simply exposing students to patterning activities result in better understanding of algebra, more qualified algebraic reasoning, higher problem solving skills, and eventually show up as better results on future TIMSS achievement tests? This was suggested in the official TIMSS report from the TIMSS 2007 test, which commented on the poor result on task M05_03 (see figure 1) with the words "Since additive changes […] are not considered particularly difficult to encode, the difficulties are probably due to lack of exposure to patterning" (Skolverket 2008, 97 author's translation). The task was solved correctly by only 16 % of Swedish students. This paper looks more closely at how some Swedish students in grade 6 approach this task in order to address questions about what these students need in future instruction concerning problem solving in general and patterning in particular. In the first analysis it was found that justification played an essential role in getting students to advance from an unsuccessful initial strategy.



In the figure, 13 matches were used to make 4 squares in a row. What is the number of squares in a row that can be made in this way using 73 matches?

Show the calculations that lead to your answer

Figure 1: TIMSS task M05_03

Patterning, problem solving and justification.

Students' difficulties with patterning appear on three levels as described by Lee (1996, 105): at "the *perceptual* level (seeing the intended pattern), at the *verbalizing* level (expressing the pattern clearly), and at the *symbolization* level (using n to represent the n^{th} array or number)." These levels align well with the three aspects of patterns mentioned in the Lgr11: how patterns are *constructed*, *described* and *expressed*. Whether these aspects are to be addressed simultaneously or separately is an open question. Some may choose to work with only the perceptual level and the construction of patterns in the early grades, verbal descriptions later, and symbolic expressions only when formal algebra has been introduced. Others may choose to work on all levels at the same time, using patterning tasks as a way of introducing algebraic notation to express generalities. Based on a Vygotskian perspective on teaching and learning, referred to as the theory of knowledge objectification (Radford 2001), algebraic thinking is described as a tangible social practice materialized in the body through gestures, visualization and perception, and in the use of signs such as words and symbols (Radford 2012). Radford shows evidence of students' use of gestures becoming part of a linguistic repertoire that helps them notice and articulate specific aspects of a pattern. In such a view on learning it is not possible to separate the levels of perception, verbalizing and expressing a pattern since all these aspects of patterning contribute to the linguistic repertoire that affords development of algebraic thinking.

Patterning activities in textbooks are commonly structured to help students generate patterns by asking them to continue the pattern, describe the next figure, describe the pattern and a create a formula for the n^{th} figure. The TIMSS task above asks students to make use of a pattern they first need to detect, which makes it a problem solving task without the scaffold of a guided step-by-step procedure. It is thus more of a true problem to solve than a didactically designed patterning task. Lee (1996) addresses the problem-solving issue of patterning using the term 'perceptual agility' to describe the ability to see several patterns in a sequence of figures or numbers and judging which patterns are useful. Such agility is closely related to justification and argumentation. Through justification a pattern will be validated and argumentation will support or refute the usefulness of different patterns.

From 1969 to 1994 Swedish curricula focused more on instruction *about* problem solving and learning *for* problem solving than learning *through* problem solving (Wyndhamn, Riesbeck and Schoultz 2000). Problem solving in the Nordic countries during this time has been described as ‘applied problem solving’ (Zimmermann, 2001). The National Syllabus (Lgr 11) treats problem solving as both core content and as an ability students should develop. Today there is a movement in Sweden to work with ‘rich problems’ emphasising open problems, different solutions and use of multiple representations (e.g. Haglund, Hedrén and Taflin 2010).

What value does problem solving and generalizing activities in school mathematics have beyond learning the specific mathematics content embedded in the problem? Zimmermann (2001, 57) lists some possible goals and characteristics of problem-oriented mathematics instruction, such as “possibilities for the invention of *conjectures* and their critical discussion, including *refutations* and *proofs*”, “*connecting thinking*” and “opportunities for *communication*”. These goals bear many similarities with the list of purposes for justification in school mathematics expressed by 12 middle school teachers in a more recent study by Staples, Bartlo and Thanheiser (2012). Justification, according to these teachers, promotes conceptual understanding and fosters mathematical skills and dispositions. One teacher says:

> Justification pushes students beyond a procedure to a deeper understanding of the math. In order to justify their thinking, they have to justify not only the hows, but get to the whys of what they’re doing. (454)

In classrooms the demand to ‘explain your thinking’ is often met by a verbalising of the procedure, whereas a demand to ‘justify your solution’ could perhaps help student develop mathematical reasoning and argumentation.

Method

Using a larger set of video data collected within the project VIDEOMAT (Kilhamn and Röj-Lindberg 2012) this study is an analysis of small group discussions when solving the TIMSS task presented above. This paper reports on 3 groups of students from 2 different grade 6 classes working on the problem. The task was given by the researchers following four lessons of introduction to algebra planned individually by each teacher as part of the normal curriculum. The aim of the intervention was to study how these students worked on the problems without specific instruction but within the context of introductory algebra. The teachers handed out and gave instructions to the group activity in slightly different ways for example concerning whether they expected individual or group documentation. To encourage discussion the final request in the original task to ‘Show the calculation’ was replaced by the question ‘How do you know?’ The group discussions were video recorded and the videos were then viewed many times by the author of this paper as well as other members of the VIDEOMAT research team. Essential parts of the interactions were transcribed. The analysis focused on students’ initial and subsequent strategies, particularly changes of strategy, as well as the nature and effect of any teacher intervention. The student groups spent 9 –14 minutes working on the problem.

Results and discussion

In this section the three groups, here called Alpha, Beta and Gamma, are presented one at a time with analytical comments.

Group Alpha (S1T1-SG2)

This group of four girls only has one paper to write on and most of the time girl C takes charge of the paper and does most of the talking. Initially A suggests drawing, B wants to work it out with numbers, and C starts drawing squares. However, C keeps losing track of how many she has drawn and the paper is too small. Her drawing is not systematic. After 2 minutes they all start discussing number facts in search of some factors for which the product is 73. They are off task for a few minutes discussing the nine times table. After five minutes B raises her hand to attract the teacher's attention and starts a new drawing, this time completing one square at a time with three lines for each. C counts as B draws, nodding her head for every count and making them very clearly in threes: *23 24 25 ,..., 32 33 34*. Then the drawing gets too small, and C takes the paper and continues the drawing coming up with ideas of how to fit more squares in. After 54 sticks the row of squares reaches the edge of the paper and B takes over starting a new row of squares. After 11 minutes both A and B raise their hands and the teacher finally joins them.

Teacher: well then what have you done?
C: we're not done, because it's too tiresome to draw all 73
B: well we don't know… Can't you like, take 73 times 4?
Teacher: what does that give you?
B: I don't know…
[A gestures that she has an idea, the teacher directs attention to her]
A: is it, can't you take, like, 73 divided by 3 minus 1. Because here, it's 3 otherwise and so if you divide those 3 and then you just take away this first one here
Teacher: yes, why did you think of that?

In the episode the teacher does not evaluate B's suggestion but asks her to clarify. Girl A, who has mainly participated as an observer, suddenly finds room to give a suggestion that shows her perception of the addition of threes and the extra one, possibly through the simultaneous drawing, nodding and counting of the others. The teacher evaluates the solution and leaves, and at the end of 14 minutes the girls hand in a paper with the solution 73/3 – 1, which is not quite correct. There is a slight call for justification by the teacher helping the girls to get past the perceptual level and beginning to verbalise the pattern, but the group never gets to the correct expression.

Group Beta (S3-SG1)

This group of two girls (B and D) and two boys (A and C) takes 9 minutes to solve the problem. Like Alpha, they start by suggesting number facts (73·4, 73/4) and then begin drawing squares, each one on their own paper. A, B and D spend the following 7 minutes drawing. A makes many drawings, rubbing them all out and starting over several times. First she loses track of how many she has drawn. Then she gets different total amounts (28 and 24). She checks both answers by multiplying 28·4 and 24·4. Neither of the products is 73 so they are slightly at a loss. C has suggested an equation but keeps seeking a multiple of 4. When dividing 13/4 he realises that 12/4=3. C exclaims that 24 squares "*feels right.*" B starts a new drawing this time systematically one square at a time counting *1234*, *567*, *8910*…They hear from another group that the answer is 24, but they are still not satisfied. After 7 minutes C calls the teacher's attention.

C: look we counted how many we could do and it is 24.
Teacher: why? [B looks up]

C: because we counted, how many, of these there are for 73
B: or, you do 72 divided by no 73 divided by 3, and then minus 1.
T: that is very difficult to calculate. In which order should you take minus and divided by? Why do you take minus 1?

In this episode, the students themselves want to find a justification, seek evaluation from the teacher who answers by requesting a justification that goes beyond their drawing. B who has been totally engrossed in her drawing suddenly looks up and finds that she has seen the pattern and verbalises it. Again the teacher questions her answer, asking her to explain why she takes minus 1 so that she can work out in which order the subtraction and division need to be done. When this is resolved, they compare 'solution by drawing' with 'solution by the general expression'.

Group Gamma (S3-SG2)

The third group is a dysfunctional group with four members. Girl A works on the task for 12 minutes, at times joined by boy B. C and D are mostly off task or trying to copy what A is writing but never contributing with ideas. As in both the other groups an initial strategy is to use number facts (13·4). Then A suggests an equation and writes $3x$=73, showing that she has perceived the pattern of multiples of three. She calculates 73/3=24,3333… and the others copy her answer. They seem to have finished when the teacher comes past. When seeing their answer, she questions the result asking if it will not be an even number of squares, or a strange sort of square at the end, one is not closed. Girl A laughs, and the teacher points to the picture asking "*what about the first square?*" She leaves them to try again. Peers from another group come by, telling them to divide 72 by 3 instead of 73 by 3. Girl A contemplates this, wondering why. After some time she exclaims: "*wait, you take it away to use at the end don't you!*" She has now perceived the pattern and expressed it verbally. She finally expresses the answer as "*3x=72 and then +1*", and explains her solution to B. In this episode the call for justification comes from the teacher when she evaluates the initial solution and questions its validity. Also A herself feels a need for justification once she knows the correct solution, and she is not satisfied until she can express clearly why she has to take off an extra stick to add at the end.

Conclusions

A summary of the analysis shows that initial, but unsatisfactory, strategies were similar in all groups. These were: using number facts, drawing squares, or writing an equation. In the process of drawing systematically, in combination with gestures and/or counting in 3's, the pattern was perceived, but not readily verbalised. In each group there was a turning point initiated by a call for justification of their first effort. This call for justification came from the teacher or from students themselves. In groups Alpha and Beta the teacher played a role of changing the pattern of participation slightly, thus giving new ideas opportunity to surface. In group Alpha the students finished when the teacher no longer asked for a justification, and therefore the problem was not fully solved on the symbolisation level, whereas group Beta were asked to justify also the order of operation and the reason for subtracting 1.

Undoubtedly, a massive exposure to similar additive pattern tasks would probably result in better average achievement, but it might not make students better equipped to solve slightly different problems. These findings suggest that students would benefit from engaging in problem solving where the *justification of a solution* becomes an essential part of the process. In addition to Polya's four stages of problem solving (Polya, 1990/1945), a fifth stage of *justification* and *argumentation* could be added. The teacher's role in small group problem solving activities is not so much to guide students in a step-by-step procedure, or to evaluate their solutions, but rather to ask them a) to justify what they are doing, b) to create opportunities for new ideas to come forward and c) to expect valid mathematical argumentation.

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