Continuous and discrete knowledge: analysing trainee teachers' mathematical content knowledge change through 'knowledge maps'

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Shulman is renowned for shifting the focus of teacher knowledge research onto content knowledge for teaching with the introduction of his categories of content knowledge. Following Shulman, many researchers have defined further categories of knowledge for teaching or refined his ideas (e.g. Deborah Ball and colleagues). Many accept that there is a specialised knowledge of mathematics for teaching. However, others argue that teaching is simply utilising mathematical content and processes within a different (teaching) context, rendering categories of knowledge types unnecessary (e.g. Anne Watson). Both points of view are taken into account in the introduction of 'continuous' and 'discrete' knowledge – a proposed metaphor for how mathematical content knowledge is held within teachers' minds. Not only do these terms aim to reconcile these seemingly opposing perspectives, but they take into account the dynamic nature of knowledge, allowing it to be represented in the form of 'knowledge maps' for comparison over time. This paper introduces the proposed metaphor and representation as a means to research trainee teachers' mathematical content knowledge change.

Keywords: mathematical content knowledge, secondary teachers, categories of knowledge, Shulman

Background

Many researchers have recognised that through teaching, teachers themselves gain a deeper understanding of mathematical content as teachers better understand both content and possible methods for teaching through their teaching practice (Leikin, Berman, and Zaslavsky 2000). Yet, despite teacher knowledge being a focus of mathematics education research within the last few decades (Hill, Schilling, and Ball 2004), "... our understanding of what and how changes in teachers' mathematical knowledge through teaching is relatively limited" (Leikin 2005).

This paper is part of a wider research project which aims to address this gap in the literature by examining possible factors which cause change in knowledge as trainee secondary mathematics teachers in England participate in teaching. However, before a change in knowledge can be analysed, a precise understanding of what is meant by 'knowledge' is required. This is the focus of this paper.

Mathematical Knowledge for Teaching

Shulman states: "A conceptual analysis of knowledge for teachers would necessarily be based on a framework for classifying both the domains and categories of teacher knowledge, on the one hand, and the forms for representing that knowledge, on the other" (1986, 10). Thus, existing categories and representations of knowledge presented within the literature are considered.

Knowledge categories:

Shulman's seminal paper (1986) reframed research into teachers' knowledge with a new focus on the role of content for teaching (Ball, Thames, and Phelps 2008). Within the literature, others have attempted to categorise knowledge for teaching either by presenting alternative categories (e.g. Leinhardt and Smith 1984; Prestage and Perks 2001) or building upon the work of Shulman. For example, Deborah Ball and colleagues further divide Shulman's (1986) 'subject content knowledge' into 'common' and 'specialised' content knowledge. 'Common content knowledge' (CCK) is mathematics knowledge which any well-educated adult should know, whereas 'specialised content knowledge' (SCK) is knowledge which is mathematical in nature and which is beyond that expected of a well-educated adult but not yet requiring any knowledge of students or of teaching.

Despite different categorisation systems and labels, many researchers recognise knowledge required *for teaching* mathematics as different to knowledge of mathematics. Conversely, (Watson 2008)argues against the use of categories of mathematical knowledge for teaching, reasoning that people have developed into effective teachers without such categories and: "...the tasks of teaching can be seen as particular contextual applications of mathematical modes of enquiry" (2008, 1).

Ball and colleagues maintain the need for categories of knowledge. This view is supported by studies which show that mathematical knowledge does not relate to effective teaching (Tennant 2006; Stevenson 2008), suggesting that knowledge required to teach is different to mathematical knowledge, (thus categories distinguishing between knowledge types can be seen as helpful). Moreover, building upon their definitions of common and specialised teacher knowledge, (Hill, Schilling, and Ball 2004) developed over 100 multiple-choice survey items to test such knowledge. The results suggested that teachers' knowledge of mathematics for teaching is "at least partly domain-specific, rather than simply related to a general factor such as overall intelligence, mathematics or teaching ability" (Hill, Schilling, and Ball 2004).

(Hill, Schilling, and Ball 2004) state that a way to distinguish between items designed to test CCK and those designed to test SCK is to imagine how a person knowledgeable in mathematics but who has never taught mathematics to students may respond to the questions. Having a degree in mathematics but having never taught mathematics to a class of students, I certainly found this a useful way to distinguish as I attempted some of the released items (see Ball and Hill 2008). Indeed, I was "surprised, slowed, or even halted by the mathematics-as-used-in-teaching items" (Hill, Schilling, and Ball 2004), whereas, I "[did] not find the items that tap ordinary subject-matter knowledge difficult" (Hill, Schilling, and Ball 2004, 16). Nevertheless, does this mean I do not hold the type of knowledge needed for teaching, or that I am simply not practiced in utilising my mathematical knowledge within a teaching setting?

There are convincing arguments both for and against categories of knowledge. In my opinion, both sides of the argument can be satisfied by considering, not categories of knowledge, but *how* that knowledge is held. On one hand, the categories presented by researchers simply offer labels to aspects of knowledge, or means to 'map out' the terrain of teacher knowledge - they label content knowledge for teaching as different to mathematical knowledge, yet they do not explain exactly *how* (thus categories are unhelpful). On the other hand, the studies used to support categories of knowledge (which show a lack of correlation between mathematics qualifications and effective teaching), can be explained by knowledge being held differently. Indeed, "Without understanding more about *how* mathematical knowledge is brought to bear on the tasks of teaching, descriptions and audits of necessary knowledge are hypothetical" (Watson and Barton 2011). In other words, if mathematical knowledge needs to be held in a different way for effective teaching, then of course mathematical qualifications do not correlate with effective teaching. Indeed: “It is not just a question of what teachers know, but how they know it, how they are aware of it, how they use it and how they exemplify it” (Watson and Barton 2011).

Forms of knowledge (how knowledge is held in the mind)

Within the literature, many have attempted to distinguish between different forms of knowledge, for example (Leinhardt and Smith 1984) introduce ‘declarative’ and ‘procedural’ knowledge; Shulman (1986) proposes ‘propositional’, ‘case’ and ‘strategic’ knowledge; and (Skemp 1976) distinguishes between ‘instrumental’ and ‘relational’ understanding.

However, there are criticisms of these terms, they are said to be: too static (Fennema and Franke 1992 cited in Petrou and Goulding 2011), not detailed enough (Prestage and Perks 2001), and can be said to be ‘dualistic’ (Tomlinson 1999). Moreover, Adler and Ball ask: “Where are different terms being used for the same ideas?” (2009, 3). In my opinion, some of the terms above are simply different labels for the same ideas.

Discrete and Continuous Knowledge

It appears that a main distinction made within the literature, although not termed as such, is between knowledge which is ‘discrete’ and knowledge which is ‘continuous’. In other words, a ‘discrete’ form of knowledge can be learned once (unless forgotten) and can be used to solve mathematical tasks in a learned, fixed way - a procedural, propositional or an instrumental knowledge. In contrast, a ‘continuous’ form of knowledge is not necessarily learned only once but the knowledge can continuously be extended, developed and connected with other knowledge - a conceptual, strategic or relational knowledge. As an example, the knowledge needed to be able to complete the square to solve a quadratic equation only needs to be learned once (unless the procedure/algorithm is forgotten). Thus one either possesses the knowledge to complete the square, or one does not at any given time. Further the knowledge of the procedure cannot be altered, deepened or extended; the process is always the same for completing the square (though of course, one can become more adept or speedy at completing the square over time). Conversely, the knowledge needed to understand quadratic equations can be extended and deepened over time as one learns further techniques for solving, connections between them, which technique would be most useful for a given equation and why the techniques work.

The discrete/continuous categorisation, whilst taking into account the differences between knowledge as highlighted (and labelled) by other researchers, emphasises the dynamic nature of knowledge by considering it in light of its behaviour over time. I purport that such a category can help overcome the limitations of other static classifications of knowledge presented within the literature and is not a ‘dualistic’ category (Tomlinson 1999). Moreover, a categorisation which focuses on how knowledge alters over time is crucial for the current research which considers teachers’ knowledge change.

Representations of knowledge

Within the literature there have been attempts to 'map-out' or represent how knowledge is organised within one's mind. For example, Ma's (1999) 'knowledge packages', and use of concept maps (e.g., Chinnappan and Lawson 2005). However, 'knowledge packages' have been criticised as being not flexible enough to be used in the practice of teaching (Ball and Bass 2000).

Taking these ideas as a starting point, a representation of knowledge termed a 'knowledge map' is proposed. 'Knowledge maps' incorporate the discrete and continuous knowledge types and also Duval's ideas of representation. He states:

> Mathematical objects, in contrast to phenomena of astronomy, physics, chemistry, biology, etc., are never accessible by perception or by instruments (microscopes, telescopes, measurement apparatus). The only way to have access to them and deal with them is using signs and semiotic representations (2006, 107).

Taking multiplication as an example, according to Duval, it is not a concept that can be perceived or measured; it is only accessed via it representations such as '×'. Extending this idea, drawing upon the multiple representations or metaphors for multiplication which were listed by participants in a study by (Davis and Simmt 2006), I present an example of a knowledge map for the concept of multiplication (Figure 1).

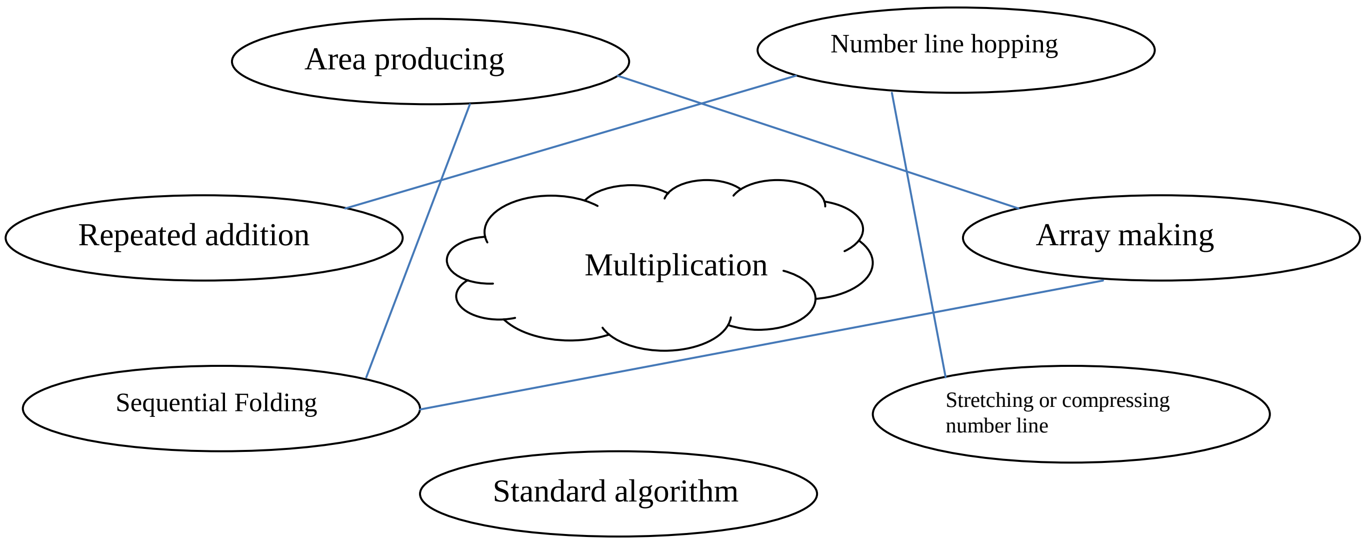

Figure 1: Example of a 'knowledge map'

Here, the central 'cloud' represents the concept of multiplication which itself cannot be perceived (grasped hold of). In contrast, the circles are discrete, concrete, ways of representing the concept or of carrying out the process of multiplication.

The addition of knowledge of further discrete representations of a concept such as multiplication is thus continuous knowledge. This can be developed over the course of teaching and can help explain how learning through teaching occurs. Such representations can be learned from text-books, students, professional development courses, colleagues and other resources teachers use and encounter during their teaching practice. Each discrete representation adds to an increasingly complex set of connections and builds the teachers' knowledge of a mathematical concept which is otherwise not perceivable. Moreover, such an understanding as represented above is not an isolated part of knowledge, but is connected to other concepts such as addition, division and square numbers.

The connections between the nodes of discrete knowledge are complex and are thus only a few possible connections are drawn on the diagram. Unlike concept maps, the concept itself (multiplication) is not a node, but the multiple representations of the concept form a nucleus of nodes with multiple connections between them and other nuclei of nodes (concepts). A ‘knowledge map’ is thus not a ‘concept map’ since it is not composed of concepts, only representations of concepts. Similarly, a ‘knowledge map’ is not a ‘knowledge package’ since the latter suggests knowledge which is pre-parcelled to take into the classroom. Rather, continuous knowledge is similar to a ‘map’ from which alternative routes (and even additional ‘landmarks’) can be highlighted/ discovered *during* the practice of teaching.

It follows that the compiling of multiple discrete representations of a concept builds understanding of the concept. For example, the concept of multiplication is more than being able to carry out the standard multiplication algorithm. Indeed: “Being able to calculate in multiple ways means that one has transcended the formality of an algorithm and reached the essence of the numerical operations - the underlying mathematical ideas and principles” (Ma 1999), which resonates with Shulman’s ‘strategic knowledge’. Understanding concepts in this way is unique to teachers whose job it is to facilitate their students’ understanding of the concept. Taking accountancy as an example of another profession which utilise mathematics, being able to carry out a multiplication and the answer are likely to be the main foci rather than knowing multiple ways to represent multiplication.

Conclusion

Knowledge maps, which draw on the metaphors of ‘discrete’ and ‘continuous’ knowledge, overcome some of the limitations of existing representations as discussed above. Further, as well as recognising the dynamic nature of knowledge, knowledge maps offer a suggestion for how the mathematical knowledge needed for teaching may be different to other professions which utilize mathematics. Thus, knowledge maps have potential to be used to represent trainee teachers’ mathematical knowledge both before and after a teacher training course in order to compare knowledge change.

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