Problem-Solving in Undergraduate Mathematics

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This report outlines two projects in problem-solving funded by the National HE STEM programme. The first project is a collaboration between the University of Birmingham and **sigma**. The *Mathematical Problem-Solving Project* (MaPS) is led by Trevor Hawkes at Coventry University and Chris Sangwin at Birmingham. Matthew Badger is the project's full-time research assistant based at Coventry. The second project is a collaboration between Liverpool Hope University and the NRICH Project based at the University of Cambridge. This project, *Problem Solving in Undergraduate Mathematics* (PSUM), is headed by Sue Pope with Nick Almond and Anesa Hosein at LHU, and Vicky Neale at NRICH. The aim of each of these projects is to satisfy the recommendations of the HE Mathematics Curriculum Summit that pertain to problem-solving. This report gives a brief overview of the aims of the project, its current state and its future direction.

Keywords: Problem-solving, undergraduate mathematics

Introduction

The HE Mathematics Curriculum Summit was held at the University of Birmingham in January 2011 (Rowlett, 2011). Operated by the Maths, Stats and OR (MSOR) Network as part of the National HE STEM Programme's Mathematical Sciences HE Curriculum Innovation Project, the Summit brought together Heads of Departments or their representatives from 26 university mathematics departments in England and Wales. Education representatives from the Institute of Mathematics and its Applications, the Royal Statistical Society, the Operational Research Society and the Council for the Mathematical Sciences, members of the National HE STEM Programme, **sigma** and the MSOR Network, and several individuals also attended.

The summit's ultimate aim was to produce a set of recommendations for targeting financial support in mathematics degree programmes for curriculum development. These recommendations were the result of the debates and discussions held therein; beginning with a debate between Alexandre Borovik of the University of Manchester and Jon McLeone of Wolfram Research on the notion that "memory, subject knowledge and technical fluency remain vital for undergraduate mathematicians in the digital age". Thereafter three discussion groups were held to address the following three points –

1. 'We can't let them graduate unless....'
2. 'If maths students can't communicate in writing or speak in public – is that my problem?'
3. 'If most maths graduates "aren't confident" in handling unfamiliar problems – should we care?'

The summit then heard and discussed presentations from Prof Jeremy Levesley of the University of Leicester on "Taking control of the assessment agenda" and Dr Neil Challis on "What do the students think about their maths degrees?" The summit ended with breakout sessions tasked with creating the recommendations for targeting financial support for curriculum development. Of the summit's 14 recommendations, the first three pertained directly to problem-solving, namely –

1. *Sharing good practice*: collecting case studies on embedding problem-solving in undergraduate mathematics degree programmes, and developing a good practice guide for teaching and assessing problem-solving.
2. *Developing a bank of problems with solutions and extensions*: these problems would allow teachers looking to implement problem-solving in their degree programmes access to robust and unfamiliar problems with which to teach students.
3. *Development of a collection of teaching resources on the development of mathematics:* these would help to develop students' awareness of the culture of mathematics, and dispel the idea that mathematics is a static and complete body of knowledge.

As a result of these recommendations departments were called to tender for £56,000 of funding for appropriate projects to address these points. The two projects of the University of Birmingham and **sigma**, and Liverpool Hope University and NRICH, were each awarded a share of these funds.

University of Birmingham and sigma

Chris Sangwin at the University of Birmingham and Trevor Hawkes at **sigma** (based at Coventry University) created the first project, to which the majority of the available funding was awarded, with which they were able to appoint a full-time research assistant, Matthew Badger. This project, later titled the *Mathematical Problem-Solving Project* (MaPS), was to complete the following tasks:

- Through consultation with colleagues in HE and beyond, to
  - find out what works well in practice
  - produce case studies of problem-solving activities undertaken by English HE mathematics departments,
  - evaluate techniques for teaching problem-solving to improve students' competence and confidence
  - investigate how these techniques can support other activities, such as modelling, projects, and collaborative group work
  - examine the nature of problem-solving itself
- Write and publish a concise 'good practice' guide with advice on access, delivery and assessment of problems
- Create collections of problem exemplars with commentaries, problems, solutions and extensions for various users, some aimed at typical first-year undergraduate modules, some designed for general consumption and structured by area and difficulty
- Provide effective and sustainable access to these and other problem-solving resources

Departmental Survey

The project conducted a survey of problem-solving teaching in university mathematics departments in England and Wales. The aims of the survey were as follows:

1. Determine which departments are teaching problem-solving in either optional or compulsory modules.
2. Discover the various approaches that those departments take to teaching problem-solving.
3. Find out whether departments explicitly mention problem-solving in their degree programmes' literature.
4. Identify candidates for case studies.

In the survey, problem-solving was defined as follows –

> 'Problem-solving' is difficult to tie down with a rigorous definition. For our purposes we mean any substantial task or activity that calls for original, lateral or creative thinking by students; brings several ideas or techniques together in a surprising way; introduces something new; illuminates some topic, e.g. with a helpful counterexample.

The survey was sent to the 60 heads of departments of mathematical sciences in England and Wales that offered a BSc in mathematics. There have been 35 responses so far, and we are continuing to chase those departments from which we are yet to hear. The survey highlighted a dearth of problem-solving teaching; with most departments either not offering a problem-solving module, or conflating mathematical modelling with problem-solving; others combined numerical analysis or operational research with problem-solving. In a number of cases it was argued that problem-solving was implicit, and given its importance in mathematics that it appeared throughout the discipline. While this may be clear to teachers of mathematics, its being part of the 'hidden curriculum' (Margolis 2001) may prevent all but a few students from becoming aware of it.

Preliminary Survey Results

The first question in the survey was "Does your institution offer a module in any of its Mathematics Degree Programmes which requires students to engage in problem-solving?"; the responses were as follows –

Answer	Number of Responses
No	12
Yes, an optional module	5
Yes, a compulsory module	18

If the response to this question was either variety of 'Yes', further questions were asked about the way the module was taught, the point in the undergraduate programme where it lay, and for how long it had been running. However, from the responses to these later questions, it was clear that those who answered in the affirmative frequently did not meet our definition of problem-solving. From the descriptions given, eight responses were judged to be count as modelling, nine were neither problem-solving nor modelling, and only six remaining modules satisfied our definition of 'problem-solving'. These initial responses were used to identify case-studies. The complete results will be included in the good practice guide.

Other work

Besides the survey, the project is progressing in a number of areas:

- Work on the 'good practice' guide is in progress, and will be divided in to two main sections:
    - *Theory* – this section gives an overview of the pedagogies that currently inform problem-solving teaching in England and Wales. It draws as its main influences the work of Pólya (1990) and Mason with Burton and Stacey (1985).
    - *Practice* – the second main section expounds the ways in which problem-solving may be taught alongside traditional lecture modules in mathematics departments at undergraduate level. Here the case studies are used as the basis for the range of approaches to teaching.
- Invited essays:
    - John Mason is writing an essay on the relevance of Pólya's work in the 21st Century, and we are seeking authors for other essays in related areas.
- Case studies:
    - Six mathematics departments' ventures in teaching students with problem-solving are being reported on to inform the practice section of the guide; each case study will be included in the appendix. The universities of Birmingham, Durham, Leicester, Manchester, Queen Mary and Warwick are the subjects of the case studies, the writing of which are in progress.
- Website:
    - Matthew Badger is creating the MaPS Project website. This is currently in development and was demonstrated at the BSRLM day meeting in Manchester in March 2012.

Liverpool Hope University and NRICH

There are two main aims of this collaboration

- To develop an innovative and sustainable online bank of starting points for problem-solving, presented in an interactive, visual and engaging way that will nurture mathematical thinking, logical processes and modelling. The starting points will permit a range of teaching approaches – individual, small group and whole class. They will be fully functional on a range of digital technologies including handhelds.
- To contribute to the guide for incorporating problems into courses, developing problem-solving skills and assessment and providing case studies of effective integration of problem-solving into courses.

A programmer has been contracted to develop four starting points for problem-solving: Picture This!, an environment for exploring graphs, linear programming and filling geometric objects. These are being trialled with undergraduate students and refined to ensure they are robust on different platforms and can be used effectively by individuals, small groups or with a class, as suits different lecturers.

Ultimately they will be hosted by NRICH and accessible via its website. NRICH's considerable expertise in making starting points available for mathematical problem-solving has informed the way in which the problems have been developed.

In 1980 Papert set out a vision for how computers might transform the way in which mathematics is learnt. Almost 20 years ago BecTa (1993) declared that technology allows learners to explore mathematics in a way that enables them to –

- *Learn from feedback:* The computer often provides fast and reliable feedback that is non-judgemental and impartial. This can encourage students to make their own conjectures and to test out and modify their ideas.
- *Observe patterns:* The speed of computers and calculators enables students to produce many examples when exploring mathematical problems. This supports their observation of patterns and the making and justifying of generalisations.
- *See connections:* The computer enables formulae, tables of numbers and graphs to be linked readily. Changing one representation and seeing changes in the others helps students to understand the connections between them.
- *Work with dynamic images:* Students can use computers to manipulate diagrams dynamically. This encourages them to visualise the geometry as they generate their own mental images.
- *Explore data:* Computers enable students to work with real data that can be represented in a variety of ways. This supports interpretation and analysis.
- *'Teach' the computer:* When students design an algorithm (a set of instructions) to make a computer achieve a particular result, they are compelled to express their commands unambiguously and in the correct order; they make their thinking explicit as they refine their ideas.

The way the starting points have been designed, students can explore the situation and collect information that requires them to make choices about which technologies (including pencil and paper) to use in order to solve the problem. The use of technology to support learning and provide a context within which students can explore mathematics and develop understanding has been much researched (e.g. Hoyles and Kent (2003), Hoyles, Noss and Kent (2004), Povey and Ransom (2000)) and is not straightforward. The way in which tasks are designed and used is at least as important as using technology per se.

Experiences from the piloting and trialling of the starting points will be used to form case studies that can be incorporated into the general guidance.

Conclusion

The QAA (2007) benchmark statement for mathematics, statistics and operational research (MSOR) is clear about the importance of problem-solving in any undergraduate programme. It specifies problem-solving as a generic subject specific skill to be developed in all undergraduate courses:

> graduates will have the ability to demonstrate knowledge of key mathematical concepts and topics, both explicitly and by applying them to the solution of problems. They will be able to comprehend problems, abstract the essentials of problems and formulate them mathematically and in symbolic form, so as to facilitate their analysis and solution, and grasp how mathematical processes may be applied to them, including where appropriate an understanding that this might give only a partial solution.(15)

and includes the following standards (20-21):

Threshold: demonstrate a reasonable level of skill in comprehending problems, formulating them mathematically and obtaining solutions by appropriate methods

Typical: demonstrate skill in abstracting the essentials of problems, formulating them mathematically and obtaining solutions by appropriate methods

Early evidence from the projects indicates that HE mathematics departments will benefit from guidance and easy access to sources of problems to ensure they are providing opportunities for all their students to develop these important generic skills.

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