Movement, language and mathematics: an interplay on the journey towards confidence with formal notation

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A mixed ability group of 21 9-10 year old students were taught over a three lesson period using the software *Grid Algebra*. They gained considerable confidence with reading formal algebraic notation over this time and a key feature was the blended space created whereby the notation could be read in terms of physical movements on a grid as well as mathematical operations. Three episodes from the lessons are discussed which exemplified the changing dynamic between movement, language and mathematics.

Keywords: algebra, technology, notation.

Background

Students have many difficulties with algebra , which often relate to students trying to understand and work with formal notation. MacGregor and Stacey find that students do not find it easy to express relatively simple mathematical operations and relationships in formal notation. Expressions can be viewed as something which needs to be carried out rather than objects in their own right with concatenation resulting as a consequence . Gray and Tall defined proceptual thinking to be that which enables someone to have the flexibility to see an expression both as an object and as a process.

Wilensky (1991) suggests that the level of abstraction is not so much about the notation but about someone's relationship with that object. As such algebraic notation does not *have* to feel abstract for students as it will depend upon their relationship with it and that will initially be related to the context within which it is introduced.

Study

The study was carried out with a group of 21 mixed ability 9-10 year olds in a primary school in the West Midlands. Prior to the study the students had not met formal notation nor have they used letters within an algebraic context. They were taught over three lessons using the software *Grid Algebra*[2] mainly as a whole class with an interactive whiteboard. There were occasional pen and paper worksheets based upon the software work and on two occasions the students went into a computer room to work in pairs or individually on computer generated tasks from the software. The teaching style was one were nothing was explained, instead there was extensive use of questioning and creation of various tasks. The lessons covered number activities, the creation and reading of expressions, the introduction of letters, substitution, inverse operations and solving equations. The original aim was to see how students responded to the particular visual and kinaesthetic nature of the software and how this affected the way in which they engaged with learning formal algebraic notation and solving linear equations.

[2] *Grid Algebra* is available from the Association of Teachers of Mathematics. (http://www.atm.org.uk/shop/products/sof071.html)

It should be noted that I wore three hats in this study, that of teacher, researcher and also designer of the software.

The software

Grid Algebra is based upon a multiplication grid with the one times table in row one and two times table in row two (the grid can continue with other rows but only the first two rows are shown in Figure 1). A particular feature is that movements can be made between cells on the grid, dragging one number (or indeed letter) either horizontally or vertically. Horizontally, will result in either an addition or subtraction; vertically will be either multiplication or division (see Figure 2).

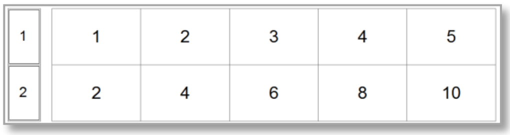

Figure 1: the grid is based on multiplication tables

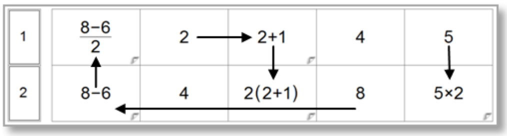

Figure 2: some movements made on the grid

Expressions can be built up through a series of movements and any particular number or expression can be rubbed out. Letters can be placed on the grid and moved as well resulting in similar notational expressions being created. There are many other features of the software but only those necessary for this paper are included.

Framework

Fauconnier and Turner talk about the notion of a blended space where two or more input spaces are blended together to make something different which helps develop a new emergent structure. Edwards offers the example of a number line which brings together the knowledge of numbers and the imagery and knowledge of a geometric line. These come together to form a number line which has properties not found in each of the others individually.

I use this notion of a blended space to consider how movement, language and mathematics come together in the students' learning of formal algebraic notation working towards the solving of linear equations. Movement concerns the journeys made on the grid using the software; language concerns both the formal notation provided by the software as a consequence of the movements made, and also the verbal language used in the classroom; and mathematics concerns the arithmetic relationship between numbers in the grid and the issue of order within an expression.

I will look at just three episodes within the three lessons. These are significant in terms of the dynamic between movement, language and mathematics. The three episodes indicate how a blended space was created and how this assisted with gaining confidence in reading formal algebraic notation. I do not attempt here to describe all aspects of the lessons nor how students ended up solving linear equations , only some key dynamics which lead to students gaining confidence with formal notation.

Episode 1: Introducing notation

The students had been involved in a number of activities which helped them become familiar with the structure of the grid but which had not involved movement. Movement was first introduced when considering the connection between numbers in the same row. This introduced addition and subtraction, such as moving from 2 to 3 in row 1 would result in 2+1 being shown in the 3 cell (see Figure 2 above). Multiplication by two was then established as a movement downwards from the 5 in row one to the 10 in row two (see Figure 2) and the class were asked what would be the opposite of this, going from the second row back to the first row. They were clear that it would be dividing by two. So, with the pen on the 8 in row two, I asked what it would say if I moved upwards to the 4 above it in row one. They replied it would be eight divided by two. I then made that movement which resulted in $\frac{8}{2}$ appearing in the 4 cell. There was a few seconds silence followed by some students saying “Fraction” or “It’s a fraction”. The students’ reaction to the sight of $\frac{8}{2}$ was indicative of the fact that they had only met the use of a division line within the context of fractions and had previously used the ‘÷’ symbol for division. This meant there was a new reading of the division line in terms of division as well as being part of a fraction. This was an example of students seeing $\frac{8}{2}$ as an object and needing also to see it as a process, which is the opposite of what is usually reported with students seeing expressions in terms of processes to be carried out and needing also to see an expression as an object . The fact that the students had said it would be eight divided by two in advance of the movement being carried out started their learning of $\frac{8}{2}$ as a way of expressing division. The mathematics was established first: what the operation would be. The notation came second so that the mathematical meaning was already present and could be placed into that notation. This created an initial conflict as they had a different meaning already established for $\frac{8}{2}$. However, the lesson continued with them being asked to chant “eight divided by two” as the relevant parts (8, dividing line and 2) were pointed to. This seemed to help them continue from then on in reading the division line as division in all future expressions.

The mathematics came first in all new movements so that students established the meaning for an expression before it appeared. Only when the meaning was established was the movement made and the notation seen. This helped also establish other notational issues such as brackets being used when multiplying an expression and the non-appearance of a multiplication sign; also the position of an addition or subtraction sign following a division.

Notation was initially read in terms of mathematical operations. However, the expressions were created through movements and as such the notation was also seen as representing particular movements. This began to set up a blended space; that of interpreting notation in terms of both mathematical operations and movements on the grid.

Episode 2: re-creating expressions

At the end of the first lesson, students went to a computer room and worked either in pairs or individually on computer generated tasks which were part of the software. These tasks involved being given an expression and having to re-create that expression through movements on the grid within a certain time period. Paulette and Sofia were working on re-creating the expression $2\left(\frac{14}{2}+1\right)$ by starting with a 14 in the second row of the grid (see Figure 3). They thought that the ‘+’ might be done first and moved to the right initially, followed by moving up. This produced the expression $\frac{14+2}{2}$. They then went back to the 14 and started again, this time producing the correct movements to get $2\left(\frac{14}{2}+1\right)$. Figure 3 shows them just beginning this correct set of movements.

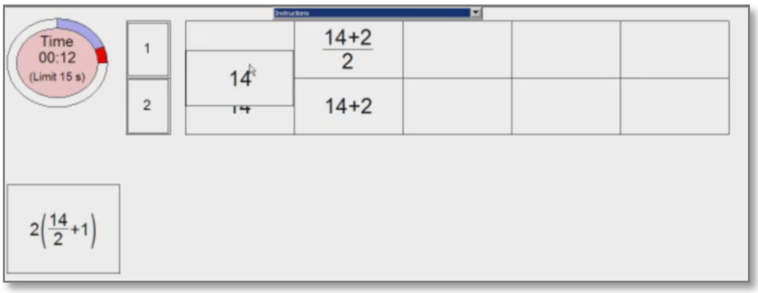

Figure 3: moving 14 up first after having previously tried moving to the right first

The expression Paulette and Sofia initially obtained looked different visually to the one they were trying to re-create. The visual feedback of the expressions created by their movements enabled them to become aware that they had not carried out the movements in the right order. This task can be viewed purely as a visual experience without the need for a particular understanding of the notation in terms of mathematical operations. The '+' sign can be read as meaning "move to the right" and the division sign as "move upwards". These signs appear as a consequence of physical movements and so can be read as arbitrary signs representing those movements. There were many such occasions when students made incorrect movements but noticed that the expression obtained through those movements was not the one they sought and this meant they re-assessed their interpretation of the order within the expression. Whether the notation was read in terms of movements or mathematical operations, students were still learning about the correct way of placing order within the notation. The fact that students engaged with the task of re-creating expressions without the need to have already a clear understanding of the notation in terms of mathematical operations enabled such tasks to be accessible for students with a range of attainment. The weakest students within the group were able to engage just as well as the most able. This meant that all students were learning to read order within expressions irrespective of what meaning they placed in those expressions; movement or mathematical operations.

This blended space allowed students to read expressions in terms of mathematical operations or in terms of movements around the grid. One student may talk about moving right and up, whilst another student talks about adding and dividing. What both students are learning and will agree upon, once successfully re-creating an expression, is the order things are carried out within that expression. One student can also think sometimes in terms of movements and at other times mathematical operations. The blended space allows all this to happen and as such allows students of a variety of abilities to engage successfully with reading order within often quite complex expressions.

Episodes 3: language shift and use of arms

When working as a whole class trying to re-create an expression it was initially common for students to use the language of movements rather than mathematical operations. After a while I began to work on shifting the language to that of mathematical operations. For example, the following transcript concerned trying to describe what had happened to produce the expression $2(10+3)-4$:

DH: Can someone again describe what I did with that journey, what that journey was. Can you describe it?

Student A: *From ten you added three, you went down the bottom and you went back four.*

DH: OK, so I added three [indicating related movement on grid with finger] and I did, what here [indicating a movement down on the grid]?

Student A: *You went down one.*

DH: OK and that is absolutely right. Is there another way of saying that?

Student B: *Times.*

DH: Don't call out please. [Indicates for another pupil to answer]

Student C: *Times two.*

DH: OK? Another way of saying that. Is that OK?

Student A: *And you're going back four.*

DH: OK and what's… well… [making movement with pen of going from 2(10+3) to 2(10+3)-4 on grid].

Student A: *Minus four.*

Later on, students were asked in small groups to discuss what happened to create the expression $2\left(\frac{2\left(\frac{2(33-2)+2}{2}+3\right)-2}{2}-1\right)-4$ which had been created on the grid using the interactive whiteboard. Although much of the language used was that of mathematical operations, the video showed extensive use of pointing and moving of arms to indicate the movements which must have taken place. Thus there was a blend of both verbal mathematical operations and physical movements to help students decide the order of operations within this expression. These now worked in parallel alongside each other.

Discussion

The dynamic between movement, language and mathematics changed at different points during the three lessons. Initially, the mathematics of arithmetic connections between numbers in the grid was significant in students developing meaning for the new notational language. The meaning for notation was developed not only in terms of mathematical operations but also in terms of movements, since it was through movements that these expressions were created. This created a blended space where the meaning of movement and/or mathematical operations could be evoked at any particular time for a given expression. The fact that students could engage in the re-creating expressions tasks purely in terms of physical movements meant that all students could learn about order within a formal expression irrespective of how confident they felt at the time about reading notation in terms of mathematical operations. Indeed, initially students tended to use the language of movement when talking about an expression. This was changed over time with explicit teaching techniques to shift the language onto that of mathematical operations. The blended space continued, however, with gestures of pointing and arm movements accompanying the verbal articulation of mathematical operations. Students exhibited considerable confidence with quite complex notation. As Wilensky stated the level of abstraction is not so much about the notation but is about the relationship someone has with what they are doing. The blended space allowed a relationship in terms of physical movements as well as mathematical operations and as such the students did not exhibit any sense of this being too abstract for them. The confidence gained with notation was a key factor in their continued learning of substituting in expressions with letters and solving linear equations, which they carried on to do within the three lessons.

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