The 'algebra as object' analogy: a view from school

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Treating algebraic symbols as objects (eg. "'a' means 'apple'") is a means of introducing elementary simplification of algebra, but causes problems further on. This current school-based research included an examination of texts still in use in the mathematics department, and interviews with mathematics teachers, year 7 pupils and then year 10 pupils asking them how they would explain, "3a + 2a = 5a" to year 7 pupils. Results included the notion that the 'algebra as object' analogy can be found in textbooks in current usage, including those recently published. Teachers knew that they were not 'supposed' to use the analogy but not always clear why, nevertheless stating methods of teaching consistent with an 'algebra as object' approach. Year 7 pupils did not explicitly refer to 'algebra as object', although some of their responses could be so interpreted. In the main, year 10 pupils used 'algebra as object' to explain simplification of algebra, with some complicated attempts to get round the limitations. Further research would look to establish whether the appearance of 'algebra as object' in pupils' thinking between year 7 and 10 is consistent and, if so, where it arises. Implications also are for on-going teacher training with alternatives to introducing such simplification.

Keywords: algebra; simplification; pedagogy.

Introduction

Consideration of the 'algebra as object' analogy, in explaining that 3a + 2a = 5a by saying that 'a' is the object 'apple', therefore this reads as '3 apples plus 2 apples makes 5 apples' has been explored by a number of authors over the years (eg. Clement 1982; French 2002; Kuchemann 1982). According to Tennant (2009), whilst the use of this analogy may get over an immediate classroom obstacle, there are three reasons for not using it:

- It's wrong, 'a' is never the object apple, in context it is likely to be either the cost or the number of apples;
- It works in only a narrow band of examples, so needs to be abandoned before reaching simplifications such as -2a + 3a and 2a × 3b;
- It leads to misconceptions and wrong answers later on, with Clement (1982) providing the classic problem: if a university has six students to every one professor, and S giving the number of students and P giving the number of professors, write down a formula connecting S and P. The distractor which arose, which the authors also have frequently obtained working with children, trainee teachers and experienced teachers, is P = 6S, which makes sense thinking of P and S as objects – for every one professor there are 6 students.

Tennant further gave a series of examples as to how early simplification of algebra might be introduced through other means, particularly keeping consideration of early algebra as close to number as possible.

This research takes the understanding of 'algebra as object' further in undertaking a small classroom based project looking to see what resources are used, and how teachers and pupils at different stages conceptualised algebra as object. This research is reported below.

Methodology

The research was undertaken in a large 11-18 comprehensive school with a significant number of pupils speaking English as an additional language. The work was undertaken in the context of a masters degree project by the principal author, a teacher at the school. The research consisted of:

- An examination of 10 textbooks and 3 electronic resources in use within the school that dealt with elementary simplification of algebra;
- Group interview of 8 mathematics teachers as to how they approached simplification of algebra;
- Individual interviews of 35 year 7 pupils asking them to solve problems of the form '3a + 2a = 5a' and then explain how they did them;
- Group interviews with 30 top set year 10 pupils based around an activity as to how they would explain early simplification of algebra to year 7 pupils.

The textbook analysis was based on Kuchemann's (1981) typology of how algebra is introduced, with teachers surveyed as to which resources they used separately to the interviews conducted.

The year 7 pupils were chosen at random, by selecting every fifth child from the school register. The year 10 class, which was not normally taught by the researcher, was allocated chosen on a convenience basis, with group interviews following.

Results

Textbook analysis

Textbooks which had been indicated by teachers as currently being in use had publication dates varying from 1955 to 2010. Of the 10 texts chosen, four went straight to algebra with no indication of context, three came through generalised arithmetic, two used symbols, effectively using 'algebra as object', and one used lengths of Cuisenaire rods. No clear patterns were established between methods of introduction and publication date.

Teacher interviews

Teachers were firstly shown a number of methods of introducing simplification of algebra from textbooks. Comments were largely based around the number of words used in an explanation and the standard of presentation rather than the explanation as such. One starting point shown to the teachers was '❀+❀= 2❀', which is taken from a revision guide (Lindsell 1998) and on the face of it is appealing to an 'algebra as object' approach. Some teachers liked the 'visual approach' and felt it would be good for low attainers, although one teacher did note that it was not 'numerical at all' and another was concerned that it might be taken to imply eg. that 4 + 4 = 24.

When asked directly about the use of the 'algebra as object' analogy, all teachers were aware that it was 'not recommended' although one teacher could not articulate why. Reasons given by others included, "You cannot multiply apples". However, approximately half the teachers involved justified the use of the approach when dealing with lower attaining pupils, particularly in lower sets in Key Stage 4. In responding to the question as to how they would approach the teaching of simplification of algebra, there was again a clear sense that different approaches needed to be taken with different year groups and attainment ranges. It was noticeable also that teachers who spoke out against 'algebra as object' methods, when asked what they did, described methods, eg. drawing pupils' faces on the whiteboard, which would appear to come under this heading. This point is discussed below.

Year 7 interviews

35 pupils were asked to simplify '3a + 2a' and then explain how they went about it. 9% of pupils said they guessed or did not know. A further 9% got the answer wrong, with answers including 5, 5aa and 7 which would appear to be consistent with the misconceptions arising from an 'algebra as object' approach.

83% of the pupils got the answer right. Of those, when asked to explain the method, 11% said 'collecting like terms', with the remainder giving an answer approximating to "2+3 then put the 'a' back on". None of the pupils explicitly stated a method which would indicate an 'algebra as object' approach.

Year 10 interviews

30 pupils working in a total of 8 groups were asked how they would explain introductory simplification of algebra to pupils. Of those 8 groups, 6 gave methods which would come under the heading of 'algebra as object', with objects including fruit, coloured objects and types of sweets. The other two groups went straight to algebra, with one stating, "It is just law".

When asked about how they would deal with expressions like 'ab', a considerable amount of confusion took place, with suggestions from those advocating a 'fruit' approach suggesting that 'a^2' could be understood as being 2 apples, and 'ab' meaning 'apple and banana', apparently confusing addition and multiplication. Those advocating a coloured object approach suggested that if 'a' represented red and 'b' represented blue then 'a×b' could be purple.

Discussion

Clear evidence arises from the research that, in the relevant school at least, the 'algebra as object' analogy is present in some textbooks in current usage, and teachers' and pupils' thinking.

The suggestion that different methods need to be used for higher and lower attainers, particularly that the 'algebra as object' with its associated problems is permissible when working with lower attainers, would stand in tension with evidence (eg. Watson 2002) that pupils deemed lower attaining, given opportunities for creative thinking, can indeed rise to the challenge. One may well consider, however, that with the constant pressure on teachers to be achieving higher and higher GCSE and other results, the temptation to resort to 'quick fix' solutions in the name of training pupils to pass examinations irrespective of underlying understanding, that this is an issue which will remain with us for the indefinite future.

That teachers of mathematics could not always articulate problems with the 'algebra as object' approach, and used themselves methods which would come under this heading, points towards the need for on-going professional training on this point as part of a larger programme of making algebra accessible and, in introduction, an extension of ideas which can be explored numerically, particularly in ensuring that methods which avoid 'algebra as object' do so comprehensively, rather than coming back to it.

A point of interest within this research is the difference between the year 7 and year 10 respondents. As noted, no year 7 pupils explicitly suggested an 'algebra as object' approach although, particularly with the wrong answers, it may well be that this was underlying their thinking. It is curious that year 10 pupils did explicitly suggest 'algebra as object', although their thinking appeared to be that the 'a' was an added complication which just needed to be manipulated and put back on at the end of the answer. This points to the need for further study in terms of pupils' understanding, as to whether this is a consistent finding in other schools and, if so, why year 10 pupils are so more attracted to the possibility. Allied to this point is the need to problematise methods of teaching simplification which do not use 'algebra as object': from a learning point of view, is it all that different, for example, to refer to "2 lots of 10 plus 3 lots of 10" rather than apples?

Conclusion

It is clear from the research reported here that there is work, both in terms of ongoing teacher training and further research, still required in considering how to address simplification of algebra in order to promote good understanding of what algebra can do. The authors would be interested to hear from anybody either with teaching ideas or interested in networking on further research.

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