Assessing young children's understanding of multiplication

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This study is part of a Nuffield Foundation-funded project aimed at developing primary children's understanding of mathematics via the use of visual representations. As part of this project, a test of primary children's understanding of multiplication was developed. Although there has been research on developing tests of understanding in other mathematical topics (e.g. fractions) and in particular for older (e.g. late primary or secondary) children, there has been little reported work on tests for multiplication for younger primary children. In this study therefore, a test of multiplication was constructed based on the range of contexts and representations associated with multiplication. The 19-item test was administered to a sample of mainly Year 3 pupils (n=272) with a small sample of Year 4 pupils (n=18) in 10 primary schools. The age of the pupils meant that test questions were read out by the teacher, and most were multiple choice items. The data obtained from the test were analysed using Rasch analysis in order to examine the reliability of the overall measure, and the validity of individual items. Two items were shown to have poor fit statistics and were excluded from the analysis. Overall, the resulting measure was shown to have a good reliability indicated by a Cronbach α value of 0.79. The analysis of the data was also used to examine the progression in difficulty of the different items, and related to the progression in children's thinking in multiplication. Recommendations for further improvements in the test are put forward.

Keywords: multiplication, assessment, primary, understanding, Rasch

Introduction

For primary school children, authors (for example Anghileri 2000; Davydov 1991; Greer 1992) have suggested that multiplication is significantly more difficult than the addition and subtraction operations. Anghileri (2000) highlights the fact that multiplication, unlike addition and subtraction, is a 'binary' operation with two distinct inputs for the multiplicand and the multiplier. Nunes and Bryant (1996) suggest that "multiplication and division represent a significant qualitative change in children's thinking" (p.144).

Another reason for the difficulty of multiplication is the range of situations in which the concept of multiplication can arise. Greer (1992) highlights a range of different 'classes of situations' for multiplication, including equal groups, equal measures, rate, multiplicative comparison, multiplicative change and Cartesian product situations. Greer (1992) also highlights the range of different external representations for these situations. Equal groups and Cartesian product situations can be represented by diagrams of equal groups of objects and arrays respectively. Skemp (1986) highlights the usefulness of the array representation in showing the commutative and distributive laws for multiplication. Anghileri (2000) also highlights the repeated groups or sets representation, with children viewing multiplication as repeated addition in their early understanding of multiplication, and then again the array which is useful for illustrating the commutative law. The number line can be used to represent multiplication (Greer 1992), particularly in relation to repeated addition. We can also represent multiplication in a symbolic way. For example, Lampert (1986) highlights the importance of 'computational knowledge', manipulating numerical symbols often according to procedural rules,

In terms of assessing understanding in multiplication, we can view understanding of a mathematical operation as the connections made between the different situations and representations of the operation (Hiebert and Carpenter 1992; Barmby et al. 2009). For example in the area of fractions, extensive work has been carried out developing assessments of understanding based on the different contexts and representations associated with fractions (e.g. Baturo 2004; Charalambous and Pitta-Pantazi 2007; Pantziara and Philippou 2011). In the case of multiplication, questions examining pupils' recognition of multiplication in different contexts (e.g. Hart 1981) or different representations (Wright, Martland and Stafford 2008) have been used. However, unlike fractions, there has been little work carried out on developing assessments specifically for pupils' understanding of multiplication. Therefore, this study describes the development of such a test, focussing particularly on the testing of younger children (Year 3 in England or aged 7-8).

Constructing the test

Based on our view of understanding, a 19 item test[1] of multiplication was constructed, drawing particularly on the research by Greer (1992) highlighting the different situations and representations for multiplication. Table 1 below relates the items to the different classes of situations.

Table 1: Relating items to different classes of multiplication situations

Class	Items
Equal groups	Q2, Q7
Equal measures	Q4
Rate	Q9, Q11
Multiplicative comparison	Q13
Multiplicative change	Q14
Cartesian product	Q16

In addition to these situations for multiplication, questions designed to probe children's computational knowledge were also included (Q5, Q6, Q8, Q10, Q12 and Q15). These included two questions (Q10 and Q12) which probed children's ability to use known multiplication facts to derive new facts.

Q10. 25 × 18 is more that 24 × 18. How much more?

Q12. If you know that 18 × 4 = 72, how would you work out 18 × 3?

Five items designed to test children's familiarity with different representations of multiplication were also included. This included an equal groups diagram (Q17), an

[1] The multiplication test is available to download from http://www.dur.ac.uk/education/research/current_research/maths/visual_rep/

array (Q18), and a number line (Q19) representation. Also included were two items (one for equal groups, one for an array) where the representations were screened – i.e. the array diagram could only be partially seen, or the equal groups items were 'hidden under cups'. These items were designed to extend children beyond simple counting of objects in the representations, and were based on items used by Wright, Martland and Stafford (2008). Finally, two items (Q1, Q3) were included as distractor items, involving subtraction and division. These items were not included in the analysis of data. The test was also referred to as a 'maths test' rather than a 'multiplication test'.

The majority (12 out of 19) of the items were multiple choice questions with four responses to choose from. Two of the computational knowledge questions were left open however, and the five representation items asked children to fill in three boxes (☐ × ☐ = ☐) to denote the calculation being shown in the picture. All of the text in the test was read out loud to children in order to take into account any possible reading difficulties that they faced. Despite this, it is acknowledged that trying to answer questions that were read out to them may have placed some strain on the working memory of pupils, particularly for more complex questions, and we will examine this issue further in the discussion.

Sample tested

The project within which this work was carried out aimed to develop Year 3 (ages 7-8) children's understanding of multiplication. Children in England are introduced to multiplication concepts in Year 2 of their schooling, therefore Year 3 is an appropriate time to test their early understanding of this mathematical concept. Children from 10 primary schools in the North East of England (some including mixed age classes including Year 4 pupils) were tested at the start of their school year of 2011/12. The schools involved had volunteered to take part in the project.

Analysis of results

The responses from the tests completed by the sample of children were marked and entered into a spreadsheet ready for further analysis. The five representation items were given marks in the range 0 to 3, according to how many boxes were correctly filled in for these items. The other items were simply marked 0 or 1 for incorrect/correct responses.

This data was then analysed using Rasch analysis. Rasch analysis is a one-parameter item response theory (IRT) model, in which the probability of a person being successful on a given item is modelled in terms of a mathematical function involving the difficulty of the item and the ability of the person (Bond and Fox 2007). This function is given by the following equation:

$$P_i(\theta) = \frac{e^{(\theta - b_i)}}{1 + e^{(\theta - b_i)}} \quad \text{(Equation 1)}$$

P_i is the probability of a candidate answering item i correctly, θ is the ability of the candidate, and b_i is the difficulty of the item. Estimation methods are used to find values for person abilities and item difficulties, and these are given on the same scale. The Rasch model can be used for dichotomous responses (e.g. right and wrong), or extended to cover more than two responses (Wright and Mok 2004) including missing responses. An underlying assumption of Rasch analysis is that the construct that is being measured is unidimensional in nature (Bond and Fox 2007). Items that do not fit the Rasch model, and may therefore be problematic for measuring the particular ability under investigation, may be identified through 'fit' statistics obtained from the Rasch analysis. In this study, WINSTEPS software was used to carry out this Rasch analysis.

Results

The Rasch analysis revealed two items with fit statistics outside the recommended range of 0.7-1.3 (Bond and Fox 2007). These were Q13 (involving multiplicative comparison) and Q16 (involving Cartesian product), and these items were subsequently removed from the analysis. The resulting set of items were calculated to have a reliability of Cronbach α = 0.79. Figure 1 shows the item-person map for the remaining items, showing the ordering of items (easier items at the bottom) relative to the abilities of the cohort of pupils sampled.

Having obtained an ability measure for each pupil, and the difficulties for the items, equation 1 was used to calculate the probability that Year 3 pupils in this sample would answer each item correctly. The small number of Year 4 pupils were excluded from this analysis.

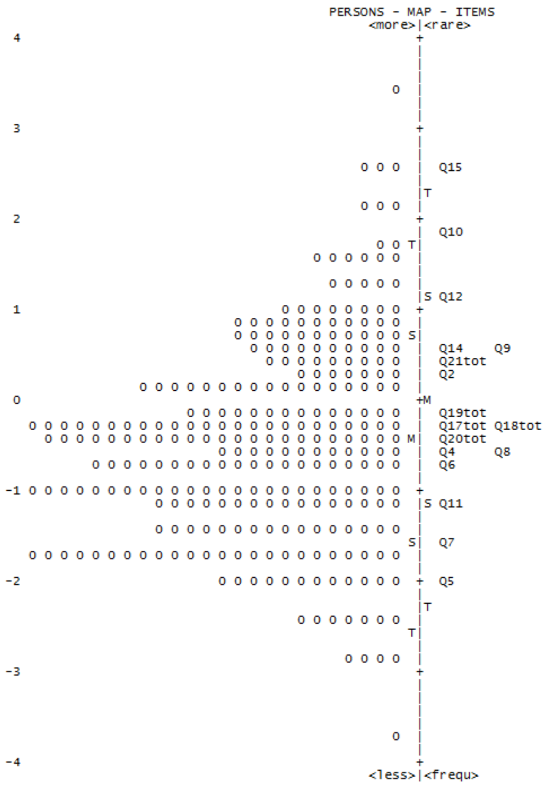

Figure 1: Item-person map for the test.

Based on the average Year 3 pupil in the sample having an ability of -0.46 logits, Table 2 gives the likelihood of correct answers for each item. Items involving multiplication with two digit numbers (Q15 – Calculate 14 × 6) using known multiplication facts to derive new facts (Q10 and Q12) were most difficult for children. Q5 (What is 5 times 2) was the easiest item for pupils. It is interesting to note that only 31% of the children were likely to answer Q2 correctly (Six children are holding two books each. How do you work out how many books they have altogether?). Also, less than half of the Y3 children could interpret representations of multiplication including equal groups, the array and the number line.

In addition to calculating the likelihood percentages, the item difficulty measures were used to examine the clustering of items in terms of difficulty. The clustering method used by Pantziara and Philippou (2011) was used to identify 8 clusters of items, based on the separation between items, and accounting for 82% of the variance of item difficulty differences. These clusters are represented in Table 2 by the dotted lines between items, and clearly reflect the clustering of items shown in

Figure 1. Other than the separation of the items at the extremes of difficulty in the measure, two clusters of items emerge: items Q6 to Q19 (total), and Q2 to Q9.

Table 2: Item difficulties and probability of correct response from sample of Year 3 pupils

Item	Item difficulty (logits)	Probability of correct response (%)
Q15	2.56	4.7
Q10	1.86	9.0
Q12	1.21	15.9
Q9	0.63	25.3
Q14	0.63	25.3
Q21tot	0.42	29.4
Q2	0.34	31.1
Q19tot	-0.19	43.4
Q18tot	-0.28	45.6
Q17tot	-0.3	46.1
Q20tot	-0.43	49.4
Q4	-0.54	52.1
Q8	-0.54	52.1
Q6	-0.71	56.3
Q11	-1.12	66.0
Q7	-1.58	75.5
Q5	-1.95	81.7

Discussion

The Rasch analysis used to analyse the data obtained from the test showed that a reliable, unidimensional measure of young children's understanding of multiplication was developed. The analysis also shows possible areas of development for the measure. Two items where the fit statistics indicated that they did not fit with the unidimensional measure were Q13 (involving multiplicative comparison) and Q16 (involving Cartesian product). One possible reason for the lack of fit for these items may have been the complexity of the question, in terms of pupils taking in the necessary information being read out to them, and the possible strain placed on pupils' working memory. Therefore, in developing this assessment further in the future, it will be necessary to assess older pupils with the measure to further validate the measure. In doing so, a greater mix of items may be required with more non-multiplication items so that pupils do not quickly identify each question being about multiplication. Also from Table 2, it is likely that only a minority of items would have been answered correctly by this Year 3 cohort. Therefore, easier items need to be included in the test for it to be more appropriate for the range of Year 3 pupils as in this cohort. Finally, looking at the clustering of items, there seems to be a progression in children's understanding, and a distinction between items, involving simpler number calculations (Q6, Q8), relatively straight forward contextualised questions (Q4) and interpreting representations (Q17-20), to more involved contextualised questions (Q2, Q9, Q14). The latter may be more complex due to the language involved (e.g. 'altogether', 'week'). The most difficult items were found to be those involving understanding of manipulation of symbolic calculations (Q10, Q12, Q15). These issues will be explored during further refinement of the test.

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References

Anghileri, J. 2000. *Teaching number sense*. London: Continuum.

Barmby, P., T. Harries, S. Higgins, and J. Suggate. 2009. The array representation and primary children's understanding and reasoning in multiplication. *Educational Studies in Mathematics* 70(3): 217-41.

Baturo, A. R. 2004. Empowering Andrea to help year 5 students construct fraction understanding. In *Proceedings 28th Annual Conference of the International Group for the Psychology of Mathematics Education*, ed. M. J. Hoines and A. B. Fugelstad, Vol. 2, 95-102. Bergen, Norway: PME.

Bond, T. G., and C. M. Fox. 2007. *Applying the Rasch model (2nd Edition)*. Mahwah, N.J.: Lawrence Erlbaum Associates.

Charalambous, C. Y., and D. Pitta-Pantazi. 2007. Drawing on a theoretical model to study students understandings of fractions. *Educational Studies in Mathematics* 64(3): 293–316.

Davydov, V. V. 1991. A psychological analysis of multiplication. In *Psychological abilities of primary school children in learning mathematics. Soviet Studies in Mathematics Education Series, Volume 6*, ed. L. P. Steffe, 1-85. Reston, VA: NCTM.

Greer, B. 1992. Multiplication and division as models of situations. In *Handbook of research on mathematics teaching and learning*, ed. D. A. Grouws, 276-295. New York: Macmillan.

Hart, K. M. 1981. *Children's understanding of mathematics: 11-16*. London: John Murray.

Hiebert, J., and T. P. Carpenter. 1992. Learning and teaching with understanding. In *Handbook of research on mathematics teaching and learning*, ed. D. A. Grouws, 65–97. New York: Macmillan.

Lampert, M. 1986. Knowing, doing, and teaching multiplication. *Cognition and Instruction* 3(4) 305-342.

Nunes T., and P. Bryant. 1996. *Children doing Mathematics*. Oxford: Blackwell Publishers.

Pantziara, M., and G. Philippou. 2011. Levels of students' "conception" of fractions. *Educational Studies in Mathematics*, in press.

Skemp, R. R. 1986. *The psychology of learning mathematics*. Harmondsworth, England: Penguin Books.

Wright, B. D., and M. M. C. Mok. 2004. An overview of the family of Rasch measurement models. In *Introduction to Rasch measurement: Theories, models and applications*, ed. E.V. Smith and R.M. Smith, 1-24. Maple Grove, MN: JAM Press.

Wright, R. J., J. Martland, and A. K. Stafford. 2008. *Early Numeracy: Assessment for teaching & intervention (2nd Edition)*. London: Sage.