Models and representations for the learning of multiplicative reasoning: Making sense using the Double Number Line

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There has been a great deal of work on the didactical use of models, such as the Double Number Line, much of it focused on using models as a support for teaching (e.g., Van den Heuvel-Panhuizen 2003). However, less attention has been devoted to documenting the ways in which students make sense of and engage with such models in developing understandings of multiplicative reasoning. In this paper, we will discuss this issue drawing on data from the ESRC-funded Increasing Competence and Confidence in Algebra and Multiplicative Structures (ICCAMS) study. Drawing on lesson observations, we will examine the relationship between the Double Number Line and students' informal methods. Our work suggests that, whilst the Double Number Line is a valuable pedagogic tool, the development of muliplicative reasoning is neverthless a long term process.

Introduction

The National Numeracy Strategy (DfEE 1999) suggests that young children should understand multiplication in terms of repeated addition and rectangular arrays, and that they should be introduced to the idea of scaling. However the Strategy offers little advice on the use of models for multiplication beyond Year 3 (age 7–8 years).

In contrast to this, extensive work on the didactical use of models, including models for multiplication, has been undertaken in Holland, from the perspective of RME (see for example Van den Heuvel-Panhuizen 2003). This work makes an interesting distinction between 'models of' and 'models for' and sees a shift from the former to the latter as a shift towards a more formal mathematical understanding.

We note also that internationally high-attaining systems such as Singapore take a more structured and sytematic approach to the use of models in the teaching of mathematics than is commonplace in England (Askew et al 2010; Ng and Lee 2009).

Given the complexity of the multiplicative conceptual field (Vergnaud 1983), we think it is important for students to develop a range of models for multiplication, including the double number line (DNL), and it is lower secondary students' encounters with this model that we consider in this paper.

Background

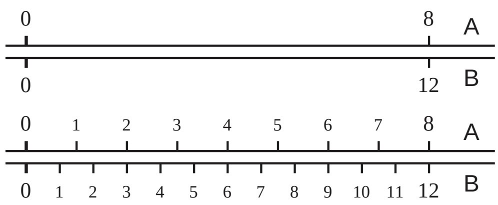
Increasing Student Competence and Confidence in Algebra and Multiplicative Structures (ICCAMS) is a 4-year research project funded by the Economic and Social Research Council in the UK (Hodgen, Küchemann, Brown, and Coe 2010). Phase 1 of the project consists of a large-scale survey of 11-14 years olds' understandings of algebra and multiplicative reasoning in England. The survey consists of two tests of multiplicative reasoning, Ratio and Decimals, together with an Algebra test and an attitudes questionnaire. In this paper, we report on Phase 2 of the study in which we are designing an intervention centred upon formative assessment (eg Hodgen and Wiliam 2006) and the extensive research evidence on the teaching and learning of algebra and multiplicative reasoning.

Phase 2 involves a total of 31 teachers from 15 schools and is focused on Year 8 (age 12-13) of secondary mathematics. Data were collected in over 100 interviews with students, from several dozen lesson observations and from tasks administered to students. In this paper, we draw mainly on data from the observations and tasks.

Models and representations of multiplication

Elsewhere we have discussed the difficulties inherent in learning multiplicative reasoning (Brown, Küchemann, and Hodgen 2010). Key to this are the different ways in which students think about multiplicative problems and the models that they use (Anghileri 1989). In England, as in many other countries, the dominant approach to multiplication is that of repeated addition. This approach has limitations including that it does not easily generalise to rationals, it does not demonstrate commutativity and it emphasises grouping over sharing approaches to division. In addition, in England, some attention is given to the use of arrays and areas to model multipication, but primarily to develop procedural techniques through the “grid method”. And whilst the area model does address some of the limitation of repeated addition, it does not easily relate to rate. Significantly most applications of multiplicative reasoning involve rate.

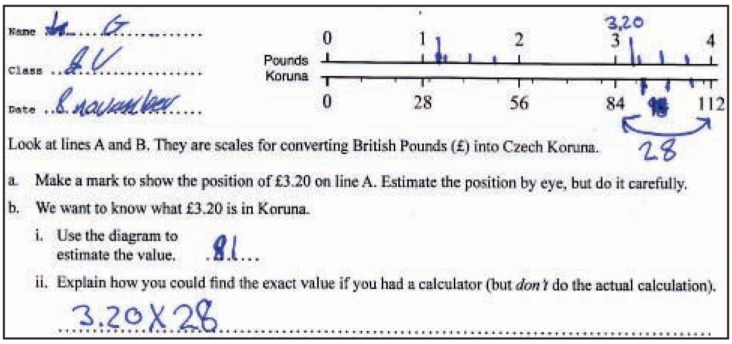
The double number line (DNL) provides a neat way of representing (and embodying) multiplication as scaling. Consider the pair of lines A and B in the top diagram (right), where 0 and 8 on line A are lined-up with (or mapped onto), 0 and 12 on line B. By drawing linear scales on each line (bottom diagram, right), *any* number (or interval) on line A is scaled by ×1.5, through being mapped onto the number (or interval) positioned directly below it on line B. A strength of this representation is that the scaling ×1.5 is brought to the fore, rather than individual object-image pairs like 8, 12. Of course, it may not initially be perceived in this way by students - the diagram can, for example, be read more concretely as a move along a number line from 0 to 12, in 8 skips of 1.5 units (where line A represents the number of skips and line B the distance skipped). However, the very fact that the DNL can model multiplication in these two distinct ways suggests that it might enable students to make links between them, and thus to develop a richer and more powerful understanding of multiplication.



Findings

To gain a sense of how students’ intuitive understanding of the DNL compared to that of the area model, we devised six tasks in which the DNL or a rectangular area were used to represent the multiplication 3.2×28 (or 3.2×2.8). Students in a quite high attaining Year 8 class were each given one of the tasks, distributed at random. The calculation was not stated explicitly in any of the tasks, but students were asked to use their representation to make an estimate and were then given this question, to see whether they could recognise that the representation was modelling multiplication:

Explain how you could find the exact value if you had a calculator.


Name ... G ...
Class ... 8V ...
Date ... 8 november ...
Pounds 0 1 2 3 4
Koruna 0 28 56 84 112
3,20
28
Look at lines A and B. They are scales for converting British Pounds (£) into Czech Koruna.
a. Make a mark to show the position of £3.20 on line A. Estimate the position by eye, but do it carefully.
b. We want to know what £3.20 is in Koruna.
i. Use the diagram to estimate the value. ...81...
ii. Explain how you could find the exact value if you had a calculator (but *don't* do the actual calculation).
...3.20X28...

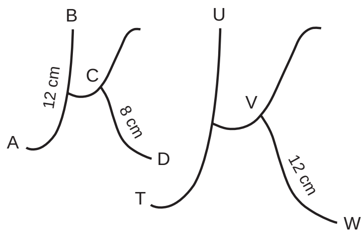
Twenty of the students were given a DNL task but only one of these students (right) gave an adequate answer to this question (3.20×28). Four other students wrote down calculations, one of which came close to giving the exact answer but involving 6 steps based on 'rated addition' (see later). Some students left this part of the task blank while others wrote "I found this value by looking" or "You can divide each sector" or "You could find the exact value by calculating".

In contrast to this, of the 12 students given one or other area task, three wrote down the calculation 3.2×2.8, two others wrote "Times the length by the width" and two wrote "Find the area".

It is perhaps not surprising that the area model is more readily seen as a model of multiplication than the DNL. However, this is not an argument for privileging one model over the other. As pointed out earlier, the models have different strengths and limitations.

Conversion (of currencies, of units of measurement, etc) and geometric enlargement both provide examples of scaling which can be modelled by the DNL. The former contexts seem relatively accessible while the latter provides a particularly rich manifestation of scaling (involving notions of similarlity and order) but is also cognitively demanding. Consider items A and B, below. They are based on items from the CSMS Ratio test (Hart 1981) and were used in a small follow-up study (Küchemann 1989) involving parallel samples of 13 to 15 year old students (N=154 and N=153 respectively). The items have almost identical numerical content, but their facilities, of 64% and 25% respectively, are clearly very different.

Item A	**Item B**
A recipe for 4 people requires 6 eggs. How many eggs are needed for 6 people?	These two letters are exactly the same shape, but one is larger than the other. How long is the curve TU?

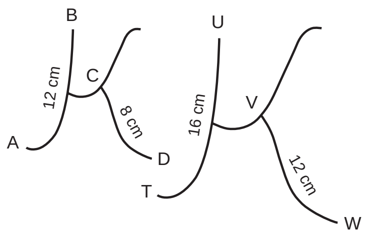


A common error with ratio items is to treat the numerical relationships additively, ie to use the *addition strategy*:

Item A: 2 more people will need 2 more eggs, making 8 eggs in all.

Item B: VW is 4cm longer than CD, so TU is 4cm longer than AB, making 16cm in all.[1]

A possible reason for the difference in facility for the two items is that students are less likely to adopt the addition strategy in item A. This turned out to be the case, with 19% of students giving such a response to item A, compared to 41% for item B. It is relatively easy to see that equating the numbers of extra people and extra eggs does not fit the given information, where the numbers of people and eggs are different. On the



other hand it may not be immediately apparent that when TU is 16cm long (right) instead of 18cm, the larger K is not similar to the smaller K.

Another common additive strategy is what has variously been called a *build up* method (Küchemann 1981; Hart 1981), *scalar decomposition* (Vergnaud 1983), or *rated addition* (Carraher 1986). Here an additive change in one variable is coordinated with a different but appropriate additive change in the other variable. Items A and B can both be solved successfully using rated addition, along these lines:

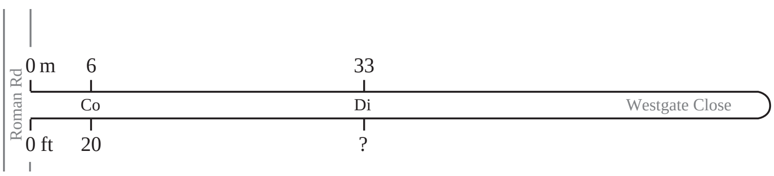
> **Item A**: 4 people + half-of-4 people will need 6 eggs + half-of-6 eggs, ie 9 eggs.
>
> **Item B**: The length of VW is the same as CD plus half of CD; the length of TU is the same as AB plus half of AB, ie 12 cm plus half-of-12cm, ie 18cm.[1]

Interestingly, this use of rated addition on Item B side-steps the issue of similarity as well as the fact that lines in the image plane have been *scaled up* rather than increased additively (scaling implies that the order of any given set of points on a line segment is preserved, which is not the case for addition). Thus the use of enlargement tasks, even when they are solved successfully, does not necessarily mean that students are engaged fully with the properties of scaling.

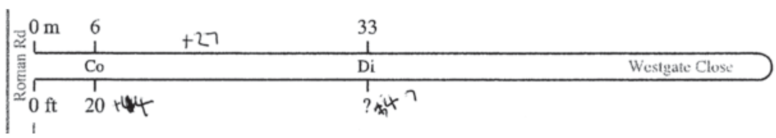
We have been investigating the use of the double number line as a way of representing scaling tasks, ie as a model *of* scaling tasks, in the hope that it might eventually serve as a model *for* the notion of scaling (see eg Van den Heuvel-Panhuizen 2003). We have developed several teaching activities, including Westgate Close (which was inspired by the imperial/metric scales used by Google Maps, and where students are given a map of a road and asked to convert lengths from metres to feet), Pounds to Leva (where students convert a sum of money), Cheesecake (where students are asked to scale the mass of an ingredient in a 100g portion of cheesecake to its mass in a 125g portion) and Stretched Ruler (where an 8cm ruler is stretched to 18cm).

The DNL provides a very concrete model in the case of the Westgate Close, which happens to be a very straight road: the drawing of the road (below) serves as the double number line and is also a very direct representation of the actual road.

Co's house is 6 m, or 20 ft, along Westgate Close.
Di's house is 33 m along Westgate Close.
Calculate Di's distance in feet.



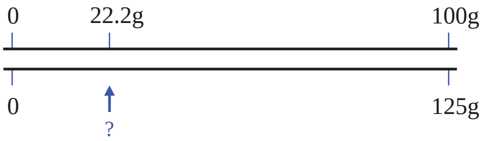
Despite this, however, some students are still be attracted to the addition strategy, as in the example, below. Here the student has found the scalar addend, 27 (6 + 27 = 33), and the functional addend, 14 (6 + 14 = 20), and then used one of these to arrive at 47 for the distance of Di's house in feet (by either calculating 20+27=47 or 33+14=47).

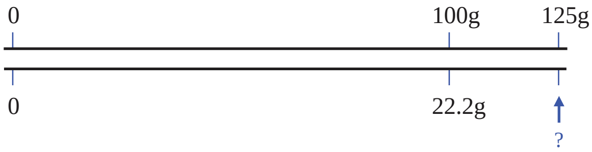


A strength of the Westgate Close context is that it is a particularly concrete context for discussing how such responses are wrong. We have found that students are generally quite good at estimating distances along Westgate Close *when asked to* *do so*. In the above example, one can do this quite easily by, for example, using rated addition and stepping up the road in intervals of 6m, 20ft, leading to 12m, 40ft, then to 18m, 60ft, etc, which clearly indicates that 47ft is far too small. One can also conduct a simple thought experiment:

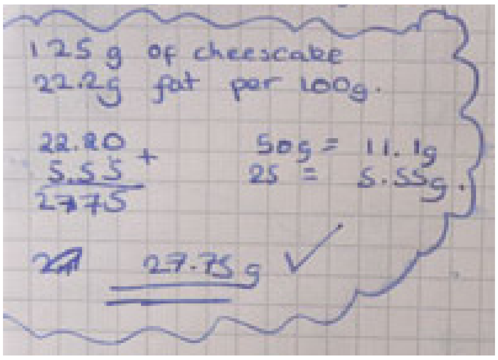
> How can adding a number of metres (27, say) be the same as adding that number of feet?

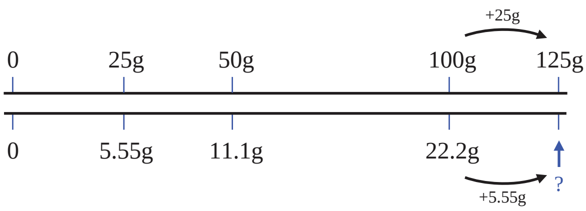
In Chesecake, both scales represent quantities measured in the same unit (grams) and this throws open the possibility of having an alternative DNL. We had envisaged using a double number line like the one on the left (below), which shows that a 100g portion of cheesecake contains 22.2g of fat, and which can be used to find the amount of fat (or any other given ingredient) in the whole 125g cheesecake. Here any number on the bottom scale is ×1.25 the number directly above it. However one can also draw a DNL like the one on the right (below), which allows one to find the amount





of fat (but only the fat) in any size portion of the cheesecake. Here any number on the bottom scale is ×0.222 the number directly above it. This scale matches the informal method used by the student whose working is shown here (right), who was understandably confused when presented with a scale like the one on the left. This student's method is very common; it involves rated addition, and, as shown below, the DNL on the right lends itself very nicely to representing the intermediate values 50g, 11.1g and 25g, 5.55g which the student used to effect the sum (100g + 25g), (22.2g + 5.55g) ie 125g, 27.75g. Just as it is possible to side-step core aspects of geometric enlargement by using rated addition, as discussed earlier, so there is a similar paradox here: while the DNL provides a model for scaling (by scaling values from one line to the other, in this case by ×0.222), it can also be used to model a non-scaling strategy, namely rated addition which involves moving *along* the number lines in tandem.





Returning to the fact that Cheesecake involves measurements in the same unit, here is a teacher's comments on the response of her Year 8 class to the DNL:

> They didn't understand this [Cheesecake DNL, like the one above, left] anywhere as easy as the Westgate Close, they were much happier with that [Westgate Close], I think, because the one was feet and one was metres, they were quite happy that these [points to double lines] were the same distance, but one was feet and one was metres, whereas this, they're both grams, but this amount of grams is different ...

In conclusion, it is clear from the foregoing that the DNL, as a model *of* scaling activites, varies in its accessibility; it also does not provide an unequivocal model *for* the notion of multiplication as scaling - though this very fact might allow students to link more concrete models of multiplication to scaling. Given the importance of the notion of multiplication as scaling, it is worth spending time developing an understanding of the DNL model, not just to solve particular tasks, but to help students, over the long term, to make a cognitive shift in their understanding of multiplication.

Note

1. This is a *between* Ks approach; it is open to debate whether this, or the equivalent *within* Ks approach, is a better match (psychologically and/or structurally) for the approach given here for Item A.

Acknowledgement

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References

Anghileri, J. 1989. An Investigation of Young ChildrenÕs Understanding of Multiplication. *Educational Studies in Mathematics*, 20(4), 367-385.

Askew, M., Hodgen, J., Hossain, S., and Bretscher, N. 2010. *Values and variables: A review of mathematics education in high-performing countries*. London: The Nuffield Foundation.

Brown, M., KŸ hemann, D., and Hodgen, J. 2010. The struggle to achieve multiplicative reasoning 11-14. In M. Joubert & P. Andrews (Eds.), *Proceedings of the Seventh British Congress of Mathematics Education (BCME7)* (Vol. 30, pp. 49-56). University of Manchester: BSRLM.

Carraher, T. 1986. Rated addition: a correct additive solution for proportion problems. In *PME 10*, London.

DfEE. 1999. *Framework for teaching mathematics from Reception to Year 8*. London: Department for Education and Employment.

Hart, K. 1981. Ratio and proportion. In Kath Hart (Ed.) *ChildrenÕs understanding of mathematics: 11-16*. London: John Murray.

Hodgen, J., KŸ hemann, D., Brown, M., and Coe, R. 2010. Multiplicative reasoning, ratio and decimals: A 30 year comparison of lower secondary studentsÕ understandings. In M. F. Pinto & T. F. Kawaski (Eds.), *Proceedings of the 34th Conference of the International Group of the Psychology of Mathematics Education* (Vol. 3, pp. 89-96). Belo Horizonte, Brazil.

Hodgen, J., and Wiliam, D. 2006. *Mathematics inside the black box*. London: NFER-Nelson.

KŸ hemann, D. 1981. Cognitive demand of secondary school mathematics items. *Educational Studies in Mathematics*, 12, 301-316.

Küchemann, D. 1989. The effect of setting and numerical content on the difficulty of ratio tasks. In G. Vergnaud et al (Eds.) *Proceedings of PME conference* (pp. 180-186). Paris, France.

Ng, S. F., and Lee, K. 2009. The Model Method: Singapore children's tool for representing and solving algebraic word problems. *Journal for Research in Mathematics Education*, 40(3), 282-313.

Van den Heuvel-Panhuizen, M. 2003. The didactical use of models in Realistic Mathematics Education. *Educational Studies in Mathematics*, 54, 9-35.

Vergnaud, G. 1983. Multiplicative structures. In R. Lesh & M. Landau (Eds.), *Acquisition of Mathematics Concepts and Processes*. London: Academic Press.