Imperative- and punctuative-operational conceptions of the equals sign

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At the British Society for Research into Learning Mathematics day conference held at the London Institute of Education in March 2011 we presented evidence for the existence of a substitutive conception of the equals sign. During the session, Jeremy Hodgen (Figure 1) questioned the use of active and passive items in the instrument, suggesting our results demonstrate not a substitutive conception but rather children's preference for passive items. This is an astute observation, and one we have investigated albeit within the context of operational rather than substitutive conceptions. Specifically, we hypothesised and tested whether Year 7 children (*N*=99) distinguished between imperative (active) and punctuative (passive) formulations of the operational conception. We found no difference, thereby refuting both our own hypothesis and Hodgen's suggestion. In this paper we present these previously unpublished findings.

Keywords: equals sign, substitutive conception

Introduction

A well defined and coherent body of literature exists on children's conceptions of the equals sign (Behr, Erlwanger, and Nichols 1976; Denmark, Barco, and Voran 1976; Kieran 1981; Knuth, Stephens, McNeil, and Alibali 2006; Li, Ding, M. M. Capraro, and R. M. Capraro 2008; McNeil et al. 2006; Molina, Castro, and Mason 2008). A key finding is that many young children, in Western countries at least, tend to view the equals sign as a "do something signal" (Behr et al. 1976), or operator, akin to "+", "×" and so on, rather than as symbolising an equivalence relationship. As children develop, many come to a relational conception of the equals sign as indicating numerical sameness, and are accepting of a wider variety of equation types. However, this happens to varying extents and even those that develop a sophisticated understanding of the equals sign readily revert to operational views of symbolic mathematics (McNeil, Rittle-Johnson, Hattikudur, and Petersen 2010). By contrast, in China the relational conception is taught from the start of schooling and this avoids many of the difficulties with arithmetic and algebra experienced by Western children (Li et al. 2008).

At BSRLM in March 2011 we presented evidence that in addition to the operational and relational conceptions there is a distinctive substitutive conception. We also demonstrated its endorsement by Chinese children but not by British children (see Jones, Inglis, Gilmore, Evans, and Dowens submitted). The evidence came from a definitions-based instrument adapted from the literature (Rittle-Johnson and Alibali 1999) in which children (*N*=243) rated the "cleverness" of fictitious definitions of the equals sign on a three-point scale. The items corresponding to relational, operational and substitutive conceptions are shown in Table 1. (Three distracter items were also included and are not shown.)

Figure 1: Jeremy Hodgen.

	The equals sign means…	Conception
R1	…the two amounts are the same	Relational
R2	…that something is equal to another thing	Relational
R3	…that both sides have the same value	Relational
O1	…the total	Operational
O2	…work out the result	Operational
O3	… the answer to the problem	Operational
S1	…the two sides can be exchanged	Substitutive
S2	…the right-side can be swapped for the left-side	Substitutive
S3	…that one side can replace the other	Substitutive

Table 1: Equals signs definitions presented at BSRLM, March 2011

Hodgen's challenge was that some items in the instrument shown in Table 1 are active while others are passive. For example, the operational items "the total" and "the answer to the problem" are passive, while "work out the result" is active. Moreover all the substitutive items suggest making an active replacement of notation, whereas all the relational items are passive descriptions. On this basis our conclusions can be called into question because children might have discerned between passive and active items rather than substitutive and relational conceptions. The distinction between passive and active items presents a valid concern, and one we have previously investigated. In this paper we present evidence that children in fact do not discern between passive and active items, and therefore the original conclusion that there exists a substitutive conception of the equals sign stands.

The operational conception of the equals sign

There are two main types of evidence that young children view the equals sign as an operator. The first is children's rejection of arithmetic equations that do not possess an expression on the left-hand side and a numerical result on the right. This is usually interpreted to mean children consider the equals sign to be an instruction to perform a calculation and write down the answer (Behr et al. 1976). The second is children's definitions of what the symbol "=" means, to which children typically respond "add the numbers", "the answer" and so on (Knuth et al. 2006).

However, on closer inspection, there are subtle inconsistencies in how scholars interpret this evidence (Jones 2008). If the symbol "=" is a "do something signal" then perhaps we should expect pupils confronted with "2+4=" to perform a calculation but pupils presented with "2+4" to do nothing. In fact this is not the case. As Behr et al.

(1976) observed, “even in the absence of the = symbol … 2+4 serves as a stimulus to do something” (13). Frieman and Lee (2004) suggested that in such cases the symbols “+” and “=” might act as a composite operator with “+” signifying the type of operation required. Some authors go further, arguing young children view the equals sign as an indicator of where the result should be written rather than as an instruction to perform a calculation. For example, Renwick (1932) observed that children “use the ‘=’ sign simply to separate an expression from its answer” (182). Similarly, Kilpatrick, Swafford, and Findell (2001) argued that for pupils “8+5 is a signal to compute” and the equals sign is “a signal to write the result of performing the operations indicated to the left of the sign” (261). Some scholars have cited children’s use of “running equations” as computational aids, as in “1+2=3+4=7”, as evidence they view the equals sign a kind of punctuation mark, performing a role analogous to a full stop in written language (Hewitt 2003; Renwick 1932; Sáenz-Ludlow and Walgamuth 1998). Sáenz-Ludlow and Walgamuth (1998, 166) argued running equations demonstrate the equals sign is “only used as an index to represent a final point of each binary state in [a child’s] over-all additive process”. Similarly, Hewitt (2003, 65) said the equals sign “breaks up the flow of left to right by creating a new beginning after [the equals sign]”.

Moreover, not all the definitional evidence quite conforms to the operational interpretation. For example, Knuth et al. (2006) asked sixth to eighth graders to define the symbol “=” and offered five responses as representative of the operational conception: “What the sum of the two numbers are” (2006, 302); “A sign connecting the answer to the problem”; “What the problem’s answer is”; “The total”; “How much the numbers added together equal.” (2006, 303). However, these arguably suggest the children viewed the equals sign as a punctuation mark than rather than as a do something signal. Similarly, McNeil and Alibali (2002) coded as operational children’s definitions of the equals sign that could as readily be interpreted as punctuative. An examination of the fictitious operational definitions we used in the study presented at BSRLM in March 2011 (Table 1) shows that one is imperative (“work out the result”) and the other two are punctuative (“the total”, “the answer to the problem”).

In light of the above we hypothesised that the operational conception might in fact be two distinct conceptions. One corresponds to viewing the equals sign as an instruction to perform an arithmetic calculation (imperative) and the other to viewing the equals sign merely as a place-indicator for writing down a number (punctuative). Evidence on this issue would also inform Hodgen’s critique that the substitutive and relational items used in our instrument are active and passive respectively.

Method

The instrument shown in Table 1 requires pupils to rate each item as “not so clever”, “sort of clever” or “very clever”. For this study the relational, substitutive and distracter items were retained, and operational items were separated into imperative and punctuative forms, shown in Table 2. Each of the three punctuative items had an imperative correlate. Items 1 and 4 both contained the term “total”, items 2 and 5 both contained “result” and items 3 and 6 both contained “answer … to the problem”.

The participants were 99 Year 7 pupils (ages 11 and 12) in a school with above average socioeconomic intake and GCSE results. The instrument was administered by the children’s mathematics teachers in class under test conditions.

Teachers were asked to encourage pupils to complete all items. Randomised hardcopies of the instrument were sent to the schools to avoid ordering effects.

	The equals sign means…	Conception
P1	…the total is written here	Punctuative
P2	…the result is found here	Punctuative
P3	…this answer is connected to this problem	Punctuative
I1	…get the total	Imperative
I2	…work out the result	Imperative
I3	…calculate the answer to the problem	Imperative

Table 2: The operational items from Table 1 separated into imperative and punctuative.

Analysis and results

The data were first prepared for analysis. Responses for each item were coded as 0, 1 or 2 for “not so clever”, “sort of clever” and “very clever” respectively. Six participants had not completed all items on the instrument and were removed from the analysis leaving a total of 93 participants. The three distracter items were removed.

To determine the children’s conceptions we subjected the data to Principle Components Analysis (PCA). The responses to the items were ordinal and accordingly we conducted PCA on a matrix of polychoric inter-item correlations (Holgado–Tello, Chacón–Moscoso, Barbero–García, and Vila–Abad 2008). First, the appropriateness of the data for PCA was checked. The Kaiser-Meyer-Olkin test produced a value of .710, exceeding the recommended .6, and Bartlett’s Test of Sphericity reached significance ($p < .001$) demonstrating the suitability of the data for analysis. We found four components explaining 27.2%, 15.2%, 12.4% and 8.1% of the variance respectively. The screeplot revealed a clear break after the fourth component and accordingly four components were extracted. The loading matrix was subjected to varimax rotation to aid interpretation. This revealed a number of strong loadings.

Relational items loaded strongly onto Component 1 and substitutive items loaded strongly onto Component 4. This confirms our previous findings that the substitutive conception exists independently of the relational and operational conceptions (Jones et al., submitted). We calculated the children’s cleverness ratings for the relational and substitutive conceptions on a scale of 0 to 6 by totalling the scores for each item. The mean relational rating, 3.04, was significantly higher than the substitutive rating, 1.68, $t(93) = 6.68$, $p < .001$, again confirming our previous findings. However, this did not address Hodgen’s critique because the substitutive items were active and the relational items passive (Table 1).

The imperative and punctuative items all loaded onto Component 2, with one exception (imperative), which was the only item that loaded onto Component 3. This refuted our hypothesis that there are two distinctive forms of the operational conception, active and passive. It also addressed Hodgen’s concern that our data in support of a substitutive conception is in fact due to children preferring passive to active items in the instrument. It is unclear why one imperative item (I6: “calculate the answer to the problem”) loaded onto Component 3 whereas the other five items loaded onto Component 2, although its passive correlate (P3: “this answer is connected to this problem”) loaded only weakly onto Component 2. A reexamination revealed I6 also loaded weakly onto Component 3 (.325).

	Predicted conception	Component loadings 1	2	3	4
R1	Relational	.855			
R2	Relational	.732			
R3	Relational	.635			
P1	Punctuative		.741		
P2	Punctuative		.637		
P3	Punctuative		.496		
I1	Imperative		.540		
I2	Imperative		.553		
I3	Imperative			.937	
S1	Substitutive				.714
S2	Substitutive				.819
S3	Substitutive				.866

Table 3: Component loadings for the items.

Discussion

Jeremy Hodgen challenged our evidence presented at BSRLM in March 2011 that a substitutive conception of the equals sign exists independently of the operational and relational conceptions. His challenge was based on the passivity of the relational items and activeness of the substitutive items used in the instrument. He proposed that we had detected a preference for passive over active formulations rather than an independent conception of the equals sign.

In this study we have presented a previously unpublished result that addresses Hodgen's challenge. Our data show that within the context of operational conceptions of the equals sign children do not discern between imperative (active) and punctuative (passive) items. As such our hypothesis that the operational conception as reported widely in the literature is an amalgam of two distinctive conceptions is refuted. Furthermore, it is reasonable to assume that this applies to all items on the instrument and that relational and substitutive items load strongly and uniquely on to different components because they genuinely measure different conceptions. We therefore conclude that Hodgen's critique, while astute, is incorrect.

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