Analysing Secondary Mathematics Teaching with the Knowledge Quartet

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This paper describes how the Knowledge Quartet (KQ), which was developed with mathematics teachers in primary schools, has been tested in a secondary mathematics context. Aspects of this research are illustrated with reference to a lesson on completing the square. First we exemplify the mapping of episodes in the lesson to the KQ, then we report how one of these episodes, concerning the choice of examples, was subsequently used in a secondary mathematics PGCE teaching session.

Keywords: secondary mathematics teaching, knowledge quartet

Introduction

A programme of research at the University of Cambridge (*SKIMA*: subject knowledge in mathematics) from 2002 to the present has investigated the mathematics content knowledge of primary trainee teachers, and the ways that this knowledge becomes visible both in their planning and in their teaching in the classroom. From this research, a framework for the observation, analysis and development of mathematics teaching was developed, with a focus on the contribution of the teacher's mathematical content knowledge. Rather than considering the generic features of the lesson (behaviour management, classroom organisation and so forth), the framework, the *Knowledge Quartet*, categorises events in mathematics lessons with particular reference to the subject matter being taught, and the mathematics-related knowledge that teachers call upon. While Shulman's distinction between subject matter knowledge and pedagogical knowledge underpins this consideration of mathematics teaching (Shulman 1986), the Knowledge Quartet (KQ) identifies situations in which such knowledge can be seen in the act of teaching. The origins of the KQ were in observations of primary mathematics teaching, and grounded theory methodology (Glaser and Strauss 1967; Strauss and Corbin 1990), in the context of one-year graduate primary teacher preparation.

From the perspective of the KQ, the knowledge and beliefs evidenced in mathematics teaching can be seen in four dimensions, named *foundation, transformation, connection* and *contingency*. Table 1 outlines these and their contributory codes which arose from grounded analysis of the primary mathematics classroom data (Rowland et al. 2009). Each dimension is composed of a small number of subcategories that we judged, after extended discussion, to be of the same or a similar nature.

Over the past seven years, there has been a process of refinement of the conceptualisation of the KQ, and enhancement of the constituent codes, both in response to additional classroom data and in the process of application. This continued in primary classrooms with early career teachers as well as trainee teachers. This paper describes how the Knowledge Quartet (KQ) has been tested in a secondary mathematics context.

Table 1: The Knowledge Quartet – dimensions and contributory codes

Dimension	Contributory codes
Foundation: knowledge and understanding of mathematics per se and of mathematics-specific pedagogy, beliefs concerning the nature of mathematics, the purposes of mathematics education, and the conditions under which students will best learn mathematics	awareness of purpose; adheres to textbook; concentration on procedures; identifying errors; overt display of subject knowledge; theoretical underpinning of pedagogy; use of mathematical terminology
Transformation: the presentation of ideas to learners in the form of analogies, illustrations, examples, explanations and demonstrations	choice of examples; choice of representation; use of instructional materials; teacher demonstration (to explain a procedure)
Connection: the sequencing of material for instruction, and an awareness of the relative cognitive demands of different topics and tasks	anticipation of complexity; decisions about sequencing; making connections between procedures; making connections between concepts; recognition of conceptual appropriateness
Contingency: the ability to make cogent, reasoned and well-informed responses to unanticipated and unplanned events	deviation from agenda; responding to students' ideas; use of opportunities; teacher insight during instruction

Research overview

Having previously considered the adequacy and relevance of the KQ to the secondary context, in 2010 we began a systematic appraisal of the KQ with secondary mathematics trainee teachers. The secondary trainees differ from their primary counterparts in two significant respects. First, the secondary trainees are all *specialist* mathematics teachers, having studied mathematics to degree level, in contrast to the *generalist* primary trainees. Secondly, the secondary trainees are supported by mathematics specialists throughout their professional placements. Additionally, the subject matter under consideration in secondary classrooms becomes more abstract and complex.

The project participants were three volunteer trainee teachers from the secondary mathematics Postgraduate Certificate of Education (PGCE) cohort at our university. Two-thirds of their 36-week course is spent working in two schools under the guidance of mathematics specialist, school-based mentors. The participants were based in different schools.

Each trainee participant taught two 'project' lessons to the same class. These were in May, towards the end of the PGCE year. Members of the research team (the authors) observed and videotaped each lesson and, after each lesson, met to undertake preliminary analysis of the videotaped lesson and to identify some key episodes in it by reference to the KQ framework. Members of the team met with the trainee to view a selection of these episodes from the lesson and for one of them to lead the discussion with the trainee in the spirit of stimulated-recall (Calderhead 1981). An audio recording was made and transcribed.

In most cases, the observation of the lesson, preliminary analysis and stimulated-recall interview all took place within two days.

In contrast with the earliest *SKIMA* research, where the framework was established, this analysis was primarily theory-driven as opposed to data-driven. The analysis consisted of fine-grained analyses of each of the lessons, both before and after the stimulated-recall interview, against the theoretical framework of the KQ.

The next section of the paper outlines one mathematics lesson taught by John, and summarises some of the episodes which were identified as mapping to the KQ framework.

John's lesson

John graduated with a BA in Mathematics and Statistics from a UK university. He was placed in an 11-18 school for his second placement on his PGCE course. For this lesson, John was teaching the second set of twelve in Year 9 (age 13-14 years). There were 27 pupils present (19 boys and 8 girls). The classroom had no interactive whiteboard but there was a data projector and laptop computer.

The focus of this lesson (planned in four parts) was to solve quadratic equations by completing the square (CTS) and to find the minimum point of a quadratic function by completing the square.

Part 1: John reminded the class about the procedure for CTS by working an example (the expression x^2+6x+8); then he gave the pupils five expressions to do themselves:
$x^2 - 8x + 14$, $x^2 + 2x - 8$, $x^2 + 6x + 5$, $x^2 + 3x - 1$, $2x^2 + 4x - 2$. Later he worked through each example on the board, drawing on suggestions from individuals.

Part 2: John demonstrated solving $x^2+8x+14=0$ by CTS, then gave them examples to solve (finding the zeros of the original expressions). While they were busy, he attempted to activate *Autograph* on the laptop, but without success. Later he 'went through' some of the examples (without technology).

Part 3: John explained sketching $y=x^2+6x+8$, finding the minimum using CTS. He set the class $y=x^2+6x+5$ to sketch as an exercise, and reviewed it later.

Conclusion: an example of sketching a quadratic function where they all contributed one piece of information towards the sketch.

Analysis of the lesson revealed that episodes could be matched to each of the dimensions of the KQ and their constituent codes; a selection of these episodes is briefly outlined below.

Foundation

John was careful with the use of language to distinguish between 'expression' and 'equation'. He emphasized that a factor of 2 can be taken out of the last expression ($2x^2 + 4x - 2$) – this arose when one pupil asked: "Do you cancel it all the way down?" (code: overt display of subject knowledge).

Transformation

In a sense, the whole lesson was about working with two quite different representations of quadratic functions – firstly symbolic and then graphical (code: choice of representation).

The six expressions that John used as examples ($x^2 + 6x + 8$, $x^2 - 8x + 14$, $x^2 + 2x - 8$, $x^2 + 6x + 5$, $x^2 + 3x - 1$, $2x^2 + 4x - 2$) were well graded and incorporated many of the possible dimensions of variation (code: choice of examples).

In part 3, for his explanation of sketching quadratics using zeros and CTS, John used the example x^2+6x+8. For a second example, he considered x^2+2x-8, hesitated, and opted for

x^2+6x+5 instead. The interview revealed his reasons for making these choices, which were rather different from the suppositions of the research team (code: choice of examples).

Contingency

The planned lesson was disrupted by the failure of the IT infrastructure in the classroom so that John did not have a graph drawing package available when he considered the graphical representation of the functions. This meant that John had to rapidly re-think his approach (code: deviation from agenda).

Connection

When sketching quadratics, John had expected to use IT to assist in eliciting the connection between (i) the zeros of the quadratic and the points where it crosses the x-axis and (ii) the form $(x-a)^2+b$ and the turning point (minimum) (code: making connections between concepts)

The (high-attaining) pupils found this problematic, though we shall never know whether this would have been the case had the technology not failed leaving John with only a whiteboard and pen (code: anticipation of complexity).

Thus John's teaching in this single lesson provided examples that correspond to each of the four dimensions of the Knowledge Quartet. The next section of the paper focuses on the six particular examples that John chose as the basis for the whole lesson and his use of terminology (emphasising *expression* and *equation*). First we will outline our analysis of John's choice along with his explanation given in the interview. This is followed by an account of how this choice became the basis of a teaching session on the secondary mathematics PGCE course.

John's choice of examples

One of the aspects that emerged from the earlier *SKIMA* research was the way in which trainee teachers chose the examples that they would use in their teaching. There were instances where this choice could give a well-graded and carefully considered progression. In other cases the examples could obscure the intention of the learning or be presented in an order which compounded the difficulty of the task (Rowland 2008).

John used six expressions as examples:

$x^2 + 6x + 8$, $x^2 - 8x + 14$, $x^2 + 2x - 8$, $x^2 + 6x + 5$, $x^2 + 3x - 1$, $2x^2 + 4x - 2$

In our preliminary analysis we noted that the examples were well-graded and incorporated many of the possible dimensions of variation (x^2 coefficient: one/not one; x coefficient: +/-, even/odd; expression factorises or not). This was one of the points that we followed up in the interview.

TR: What made you pick those six/seven examples?

John: They all have real solutions was the first thing, umm, so that when sketching them they can use the whole 'oh I have got two solutions, it crosses twice, it's a U-shape' … got one odd coefficient of x, cos they had had a bit of practice of that and if the focus was going to be on sketching there's no need to have sort of overly complicated, umm, squaring point 5s and stuff in there cos they can already do that.

Umm and the bottom example [$2x^2 + 4x – 2$] was chosen because you can take a factor of 2 out, and I thought that might be good for when we were talking about the difference between an expression and an equation; because if you are solving that you'd say 'I'll divide both sides two', whereas if you're just putting it into a completing the square form as an expression, you can't say that and so you have to take 2 out as a factor. And I think I spent a bit of time before that point, talking about. So that was where the last one came from.

TR: So if you like, you kind of invented these, with those factors in mind.

John: Yes

TR: OK, OK. So what, what made you pick on the $x^2 + 6x + 8$ for this particular …

John: That's the only one, the only one that factorises, I think. Well I know it factorises but I'm not sure if any of the others do. That factorises so that you get the solutions –1 and –5 [incorrectly!] which means that they will be able to draw the graph more easily, sketch it more easily, rather than getting a daft surd, to try and draw on their x axis, just to make the ease of the drawing on their own for the first time.

John's choice of examples was brought to the attention of the following year's cohort of secondary mathematics trainee teachers, in a PGCE subject teaching session on lesson planning and evaluation. The subject lecturer, Libby, wanted the group to consider John's use of examples and his desire to distinguish clearly between the mathematical terms 'expression' and 'equation'. The latter, along with the technique of completing the square, had both been raised as queries following the subject knowledge audit that the trainees had completed. It seemed an additional fortuitous opportunity to be able to consider each in this context.

From the post-lesson interview, we knew that John had not taken the examples from a textbook but had devised them himself the previous evening; he had thought through the examples again on his drive to school the next day, deciding that integer solutions would be quite sufficient for the processes involved.

The trainee teachers were asked to complete the square for each of John's six choices and to consider why John may have made this choice and to comment on the appropriateness of choosing such a set. They were told of John's lesson objectives outlined earlier in this paper. John's deliberate decision for an odd integer coefficient of x '*got one odd coefficient of x, cos they had had a bit of practice of that*' provided the group with the opportunity to consider both a decimal and fractional representation of halving the coefficient and the ease that each could be squared and subtracted from -1. This was the same emphasis that John gave to his pupils and was captured on video.

The final example (vi) also had the desired effect of deliberating over the non-possibility of dividing through by a common factor here but the common practice of doing so when asked to solve the resulting quadratic equations when the expressions are equated with zero (see interview dialogue above).

Conclusion

The purpose of this research was to test the 'fit' of the Knowledge Quartet to secondary mathematics teaching. The analysis of John's second lesson in this paper indicates the potential of KQ as an analytical tool in the context of novice secondary mathematics teaching. We may need to supplement the codes within existing KQ dimensions; for example, the existing four *transformation* codes might not adequately capture the kind of explanation that John had aimed for with his use of IT. So our analysis of the six lessons encourages us to pilot the use of the KQ as a developmental framework for the observation and review of lessons taught by secondary PGCE trainees during their school-based placements. This, in turn, will create yet more opportunities for testing and refining the KQ in the field.

Libby found being able to make reference to such detailed data (plan, video of lesson and post-lesson audio recorded interview) to be invaluable within a PGCE session. Discussing this lesson provided the trainees with a fine example of careful planning that was all the more powerful on account of reference to first-hand data and the fact that the teacher under scrutiny had only a year earlier been in their own position (though admittedly towards the end of his training year). Although time was spent on just one lesson, the points made are transferable in a generic sense.

Postscript

We offer the following cautionary tale about how what appeared to us to be a sensible interpretation of a part of this lesson was far from it, when we had the opportunity to follow it up in interview.

As indicated earlier, in part 3 of the lesson John explained sketching $y=x^2+6x+8$, including finding the minimum using CTS on a whiteboard. Once the graph was drawn, John considered which equation to use for the class to try on their own, suggested $y=x^2+2x-8$ (example iii) hesitated, and then opted for $y=x^2+6x+5$ (example iv). In the preliminary analysis and before there had been an opportunity to talk to John, we had come up with our own logical reasons impressed with his forethought in skipping (iii) as the next example had the same coefficient of x and would thus reduce the complexity, the dimensions of variation.

This was not the case, however. In the interview, John revealed that in the heat of the moment, he had thought (erroneously) that $y=x^2+2x-8$ would not give integer solutions and went on to one ($y=x^2+6x+5$) that he knew would. Until we pointed it out, he had not realised that the coefficient of x was the same as that in his first worked example. Indeed when it was first mentioned, John said that had he realised, he would have chosen something else. Nevertheless, after we said that we felt that retaining the $6x$ term helped with the minimum, John came back to connect with this and said, with laughter, that the students had indeed been more successful with it!

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