The Rise and Fall of Certainty

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Most people take it for granted that mathematical knowledge is certain. People believe that we can be absolutely certain that 2 + 2 = 4 or that the theorem of Pythagoras holds true on flat surfaces. However, within the philosophy of mathematics, the thesis of certainty has reached crisis point twice during the past 200 years. By examining the events surrounding these crises, I shall demonstrate the influence that philosophical accounts of the nature of mathematical knowledge have on curriculum development and pedagogy. Finally, I will examine the pedagogical implications of the leading contemporary account of mathematical epistemology.

The First Rise and Fall of Certainty

The Rise of Certainty

Euclid's *Elements,* written in about 300BCE, laid the framework for establishing the certainty of mathematical knowledge. The *Elements* provided a model for presenting pure mathematics, beginning with carefully stated axioms and definitions, followed by precise theorems and logically coherent proofs. Thus Euclid exemplified the axiomatic method, a method that "endowed geometry with a level of certainty never previously attained by any other science." (Reichenbach 1957, 1). The axiomatic method transfers the certainty of the axioms to the theorems, and conversely, the epistemological question of the truth of the theorems may be reduced to the question of the truth of the axioms. Therefore, given that the axioms are all self-evidently true, which they were alleged to be, and that the theorems all follow as logical consequences of the axioms, then, we can be certain that all the theorems are true also. Thus Euclid's *Elements* served as the paradigm for establishing truth and certainty for nearly two and a half thousand years.

In the 17th century Descartes claimed that in order to discover certainty we must begin by doubting everything, especially that which is derived through sense experiences. The only certain knowledge, he concluded, is knowledge that is arrived at by pure reasoning alone. He reasoned that the he could be certain of his own existence, "I think therefore I am." That geometry is correct, by following the reasoning laid out by Euclid. And that God exists, by a version of Anselm's ontological argument. Thus we can see that Descartes was more certain of the truth of mathematics than he was of the existence of the table in front of him.

Opposing this continental view were the British empiricists led by Berkeley, Locke, Hume and Mill. According to the empiricists all our knowledge is derived from experiences via the senses. Mathematics, which was a prime example of knowledge for the rationalists, was an exception to the rule for most of the empiricists. To account for mathematical knowledge, Hume divided all human knowledge into two categories, the relations of ideas and matters of fact. Mathematical knowledge fell under the category of the relations of ideas, the knowledge of which is arrived at through deductive reasoning and is not tainted by any uncertainties apparent in other sciences.

Kant turned Hume's fork into a more rigorous epistemological theory. He drew a distinction between *a priori* and *a posteriori* knowledge, and analytic and synthetic truths. Knowledge is *a priori* just in case it is arrived at by pure reasoning, and *a posteriori* if it is arrived at by empirical investigations. A statement is analytic if its truth can be determined by the meanings of words, and synthetic otherwise. Analytic truths are true by definition, and are therefore *a priori*, raising the question of the existence of synthetic statements that can claim *a priori* certainty. Now the theorems of geometry are, according to Kant, synthetic *a priori*. They are not merely true in virtue of the meanings of the words, the words right angled triangle, hypotenuse and other two sides, for example, do not contain any notions about the relationship of the area of the squares derived thereof. Yet, the truth or falsehood of mathematical statements can be known *a priori,* i.e. by pure reasoning alone. Kant's synthetic-*a priori* thesis cemented the certainty of mathematics into philosophical dogma, and, in an oblique reference to Quine, I call this 'the dogma of certainty'.

The philosophical belief that the certainty of mathematical statements can be arrived at through deductive reasoning meant that mathematics curricula were largely based on Euclid's *Elements*, and that mathematics lessons tended to focus on following and reproducing deductive proofs. There was a widely held belief that there was no royal road to geometry, meaning that there was no shorter path than following the expositions as laid out by Euclid. This is the way that mathematics was taught for nearly two millennia.

The Fall from Certainty

Euclid's 5^{th} postulate, the parallel postulate, was considered to be the one blemish on Euclid's otherwise perfect account of geometry. During the early part of the 19^{th} century both Bolyai and Lobatschewsky, working independently, attempted to prove the parallel postulate by contradiction. However, they discovered that no contradiction could be found by replacing the parallel postulate with another, contradictory, axiom, but that a self-consistent geometry, hyperbolic geometry*,* could be developed. Within this geometry the angle sum of any triangle is less than 180 and the ratio of the circumference to the diameter of any circle is always greater than pi, but no internal contradictions can be found. Helmholtz, following Riemann's work on manifolds, developed a third kind of geometry, elliptic geometry*,* in which there are no parallels, every pair of lines meet, the angle sum of a triangle is greater than 180 and the circumference divided by the diameter of any circle is always less than pi. Initially these new geometries were treated as curiosities, not to be taken seriously.

However, Einstein's use of non-Euclidean Riemannian manifolds as the mathematical foundation of the theory of relativity meant that the philosophical implications of the plurality of geometries had to be reconciled. The *a priori* certainty of mathematics was called into question because it would appear that the theorems of Euclid are not only uncertain, but, in the presence of massive bodies such as stars, untrue. Now if mathematics, the "cornerstone" of human knowledge, and "last bastion of certainty" turns out to be uncertain, then we may be left with "no certain knowledge at all." (Ernest 1991, 1-4) And "any uncertainty in the foundations of the most certain of all the sciences is extremely disconcerting." (Carnap 1972, 175)

The discovery/invention of the plurality of geometries led to the establishment of the Association for the Improvement of Geometry (later the Mathematical association) and the anti-Euclid vote of 1871. The Perry reform movement of the early 1900s aimed to liberalise Euclid and to free teachers from being required to follow Euclid's expositions. There was strong opposition to the anti-Euclid movement, but after Dieudonne's 'Euclid Must Go' speech at the Royaumont conference in 1959, Euclid's *Elements* was finally removed from the classroom.

The Second Rise and Fall of Certainty

The loss of certainty brought about by the discovery/invention of the plurality of geometries led mathematicians and philosophers to look elsewhere to reinstate the certainty of mathematics. Frege and Russell attempted to reduce all of mathematics to logic, and to build number theory, logically, from a set theoretic foundation. The foundationalist school grew in size and stature until well into the second half of the 20th Century, and was led in Europe by a group of mathematicians known (for quite obscure reasons) as the Bourbakis. By the 1960s the concept of 'set' was considered to be more fundamental to the whole of mathematics than the concept of 'number'.

The Inclusion of Set Theory

By the 1950s there was a growing dissatisfaction with the state of mathematics education. This was due, in part, to a desire for modernisation and a perception of falling standards. The launching of the Soviet satellite, Sputnik, in 1957, created the belief that the west was slipping behind the east in terms of technological advances, and brought mathematics education reform into the spotlight. This freed up government funding to support the reform movements that were already afoot. As Jackson explained in *The Sunday Times Colour Supplement* in 1973,

> The maths that grandmother learnt was pretty hot at working out how many kippers at a penny-three-farthings each you might get for £5. But Greek geometry and Victorian mercantile conundrums clearly didn't lead to the stars. (Moon 1986, 146)

The Royaumont conference of 1959 marked the beginning of a new paradigm in mathematics education. In the opening address Professor Stone said that "we are on the brink of important, even radical, changes in [the] mathematical curriculum." (Moon 1986, 1-5) The mood of the time was that revolution, not evolution, was required. The enthusiasm of these 'revolutionaries' with their daring and imaginative approach to shattering the traditional framework of mathematics teaching, their aim of bringing mathematics education into line with the modern developments in the subject at university level, and the promise of a new day in school mathematics soon caught on, and new maths became a global phenomenon. (Moon 1986, 43-59)

There were two distinct features to the new maths education reform. The first was in pedagogy, where the traditional classroom setting with the teacher acting as the 'sage on the stage' and the students in rows of desks performing repetitive sums was abandoned in favour of the teacher acting as a 'guide on the side' and students working, often in groups, to discover the secrets of mathematics for themselves. The second feature was the new content introduced into school curricula including set theory. Set theory was seen as the royal road to mathematics, and was claimed to not only provide *an* epistemological account of the *justification* of mathematical knowledge thereby reinstating its certainty, but it was believed to provide *the* logical foundation from which all mathematical knowledge could be *derived*.

Piaget (1973) claimed that psychological studies of the logico-mathematical development in children show that a child learns mathematics (and logic) through a progression of naturally occurring spontaneous developments. Furthermore, this sequence of developments corresponds almost exactly to the logical development of mathematics proposed by the Bourbaki foundationalists. Dienes (1960) provided guides to teachers that were typical of the era. These guides usually began with the notion of 'set' and 'belonging to a set' as a child's first mathematical concepts. They then extend this to the idea of a 'relation' and 'equivalence relation'. Next they introduced the notion of 'mappings' in general and 'one-to-one mappings' in particular. Only then did they introduce the idea of 'numbers' which were defined as the cardinality of sets.

The Exclusion of Set Theory

One of the most fundamental problems with the new maths is that to complete the programme and derive arithmetic from set theory requires an abstract system of logic involving the Peano axioms. So although set theory itself may be taught at a young age, one cannot build the rest of mathematics from that foundation until much later on. Shibata states that the modern view of mathematics "might be a royal road leading to a theoretical construction of mathematics," but warns that we "shall not be able to drive on it" without a thorough background understanding that students are unlikely to reach before university. (Shibata 1973, 266)

Even if students could complete the process, there were still serious unresolved problems with the whole foundationalist programme. Russell was all too aware of the problem of self reference inherent in his foundations, i.e. Russell's paradox, and sought ways to overcome these problems. More problematic however, was Godel's incompleteness theorem which showed that "not all the truths of mathematics can be represented as theorems in formal systems, and furthermore, the systems themselves cannot be guaranteed safe." (Ernest 1991, 11)

Paradoxes and incompleteness led many influential philosophers of mathematics to question whether certainty could be reinstated by the foundationalist approach. Influential philosophers who have rejected foundationalism, apriorism, and the certainty of mathematics include Wittgenstein, Kline, and Thom. Thom, whose address to the 1972 ICME conference caused the Bourbaki-minded French delegation to walk out in protest, called modern mathematics "an educational and philosophical error" and stated that if one teaches mathematics from set theory, "one rediscovers that there is no royal road." (Thom 1986, 75)

Thus we can see that there were strong objections to the new maths education reforms on philosophical grounds. There were also objections to the programme on pedagogical grounds. Kline (1973) published one such attack which "provoked a storm of controversy" following publication in the United States. (Moon 1986, 58) The 1970s the popular press, either fairly or unfairly, condemned new maths for its failure to maintain standards in basic numeracy. In the United Kingdom a series of publications entitled *The Black Papers* attacked the reforms. However, the articles were more damning of the liberal nature of unstructured education in general, rather than set theory in particular. As the *Times Educational Supplement* 1974 observed,

> Modern maths is probably getting it in the neck for many other things – open plan primary schools, new methods, the younger generation and the long hair that coincides with it. (Moon 1986, 149)

The movement even received prime ministerial condemnation in Prime Minister Callaghan's Ruskin College speech in 1976. Finally, with the publication of the Cockcroft report in 1982, set theory was officially banished and has now disappeared into educational oblivion.

Mathematics without Certainty

The loss of *a priori* certainty brought about by the discovery/invention of the plurality of geometries, and the subsequent failure of the foundationist programme, has led philosophers and mathematicians to interpret the very nature of mathematics in a radically new way. Ernest (1991) described the philosophy of mathematics as being "in the midst of a Kuhnian revolution," explaining that the 'absolutist paradigm', the view that mathematics is infallible and objectively and timelessly true, is being replaced by the 'falliblist paradigm', the view that mathematics is uncertain, open to revision, subjective and a product of human endeavour.

There are two distinct, but not mutually exclusive, strands to the contemporary philosophical account of the nature of mathematical knowledge. The first is quasi-empiricism, the view that mathematics is an abstraction founded on investigations carried out by manipulating and measuring medium sized physical objects. The second is social (or radical) constructivism, the view that mathematics is a human activity and that the truth of a mathematical statement is determined by the consensus of the mathematical community, rather than logical proof.

Empiricism with regard to mathematics was first put forward by Mill and is the view that mathematics is an empirical science that differs from other empirical sciences only in that its subject matter is more general and its propositions have been tested and confirmed to a greater degree. Quasi-empiricism is the weaker thesis that mathematics is an abstraction based on empirical investigations. This view is proving to be an attractive and popular alternative to apriorism, as can be seen in the very title of Lakatos' (1986) paper "A Renaissance of Empiricism in the Recent Philosophy of Mathematics." In this paper Lakatos describes empiricism as the "new vogue in the philosophy of mathematics" (Lakatos 1986, 31).

Kitcher offers perhaps the most comprehensive account of quasi-empiricism. His first premise is that mathematical knowledge is originally derived by experiences manipulating physical objects. However, Kitcher said that we do not need to have personally experienced these empirical investigations to access the knowledge derived from them. Our personal knowledge of mathematics is largely given to us by the experts of the current mathematical community, and the mathematical community received its knowledge from the previous community, and so forth all the way back to people manipulating physical objects and measuring sections of land somewhere, probably in Mesopotamia, several millennia ago. This is what Kitcher terms an 'evolutionary' theory of mathematical knowledge. (Kitcher 1984, 92)

Social (or radical) constructivists follow a sociological view of scientific knowledge, and categorise mathematics as a social practice where the truth of a mathematical theorem is not determined through deductive reasoning, but rather through the consensus of the current mathematical community. Therefore mathematical knowledge can never be deemed as certain, and timelessly true, but is constantly being revised and updated in accordance with current practice.

Therefore, the contemporary philosophical account of mathematics is that mathematics is an evolving social practice of deriving abstractions from empirical investigations. The recommended pedagogy that follows from this thesis is two-fold. Firstly, students should learn mathematics, wherever possible, by conducting hands on physical experiments, thus classrooms should be similar to the "mathematics laboratories" advocated by Kline (1972, 160). This is reminiscent of the pedagogy put forward during the new maths movement, if not the content. Secondly, the sociological nature of mathematics needs to be recognised. This need is inherent in the social constructivist's view of mathematics and also in Kitcher's evolutionary theory of mathematical knowledge, and has now been recognised in the New Curriculum which lists understanding "the rich historical and cultural roots of mathematics" as a key concept. Therefore educators should not present mathematics as an already completed, set in stone, body of knowledge, but present it as a dynamic, evolving human practise. I believe that the best way to accomplish this is to place greater focus on the context from which each piece of mathematics was originally developed, and to arrange schemes of work in historical order. Thus students may see that mathematics is a field of study that has been considered important since the dawn of civilisation, that it has grown and developed over time, that there has been drama and intrigue in its development, that real people have created/discovered mathematics, that mathematics has been studied and developed as a means to solve problems, and that it has been studied and developed for its own beauty and as an end in itself.

My final conclusion is that, for better or worse, the philosophy of mathematics drives the curriculum and informs pedagogy. We have gone from synthetic *a priori* to foundationism and now to quasi-empiricism and social constructivism, and each paradigm shift in the philosophy of mathematics has resulted in a corresponding paradigm shift in mathematics education.

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