Using ICT to develop abstraction

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As teachers we are constantly encouraged to make more use of ICT in the classroom. In tandem with this there is a perception that many students are embarking on A-level courses with little grasp of abstraction and consequently immediately struggle with the algebra. This paper attempts to use a model of conceptual development and, with the insights it provides, identify ways in which ICT can aid students' conceptual abstraction.

Keywords: abstraction; reification

Introduction

> "Mathematics at school [is] a collection of unintelligible rules which, if memorized and applied correctly, [lead] to 'the right answer'" (Skemp 1971, 3 quoted in Sfard 1991, 31).

In returning to teaching after a number of years I am struck by the unease that many A-level students have with algebraic manipulation. A majority of students appear to be embarking on A-level courses with an unsure grasp of rules that lack meaning to them, hopefully applying them to achieve the 'right answer'. It seems that their grip on abstract concepts is weak and the progress to further abstraction is destined to be built on sand.

In the main body of this article I look at a research paper by Sfard which gives a model of cognitive development explaining the mode of thinking of these students. I then give examples of how this model has helped students develop an understanding of abstract concepts.

To begin, I will give a short account of recent views on ICT and its role in teaching and learning Mathematics.

The role of ICT

The last thirty years or so has seen a wide variation in the application of computer technology to teaching (Oldknow and Taylor 2003, iii). Technology has developed at an increasing rate and so the opportunities have widened. However, there is evidence that the use of ICT in Mathematics teaching has not advanced. Indeed Ofsted reported in 2008 (2008, 27) that compared to evidence in 2001 and 2002 pupils were given less opportunity to use ICT as a tool for problem solving and "despite technological advances, the potential of ICT to enhance the learning of mathematics is too rarely realised." Furthermore, with the restrictions on the use of graphic calculators introduced in AS and A-level examinations from 2000, Ofsted reported that this had a "severely negative impact on their use as a tool for teaching and learning" (2008, 28). Thus while there are a number of tools available, including a number provided free, and technology becomes ever more available and versatile in our day to day lives, it seems that the use of ICT for the learning and teaching of mathematics is struggling to make a concomitant impact.

Models of cognitive development

Bruner

Bruner identified three modes of conceptual development: the enactive (hands-on actions), iconic (using representational images for concrete objects) and the symbolic, including words, number and logic (1966, 10 - 11). Development to the symbolic mode is facilitated by language (1966, 16) while the use of multiple representations and the ability to shift between representations both within and between modes is important for progress in cognitive development (1966, 65 - 68). The development of symbolic thought also enables "compactibility", condensing meaning into single mental objects and by use of language enabling powerful manipulation of these symbols. (1966, pp. 12 -13).

Sfard – operational and structural duality

Sfard includes a similar idea of condensing meaning into single mental objects in her explanation of the "dual nature of mathematical conceptions" (1991). Sfard describes mathematical concepts as combining both processes and objects; for instance, a mathematical function can be conceived structurally as a set of ordered pairs or operationally as computational process (1991, 5). The operational element relates to processes, algorithms, actions, while the structural element is static, "timeless" instantaneous (1991, 4). The operational side involves sequential processing, detail and is readily related to verbal expression, whereas the structural side is integrative, can be manipulated as a whole and lends itself to visualisation. Sfard describes these two elements as being complementary, "two sides of the same coin", linked in a duality in a similar way that subatomic entities act both as waves and particles. The operational, though, is regarded generally as a necessity before the structural can be achieved.

Sfard then describes the process of concept formation. This is perceived as being in three stages. The first, "interiorization", consists in "getting acquainted with the processes which will eventually give rise to a new concept (like counting which leads to natural numbers)" (1991, 18). The next stage is "condensation", at which point the process can be handled as a whole without needing to go into details. The process is thus 'squeezed' into manageable units whereby it can be combined with other processes; moreover the process might be named, enhancing its sense of being an object (1991, 19). Development of condensation would be indicated by increased ease in moving between representations. The ultimate and third stage, "reification", is when the process can be conceived as an object independent of the process from which it has evolved, emerging as "a fully-fledged object", owning distinctive properties and can become the input for other processes. Moreover, while interiorization and condensation can involve activity over a period of time, "reification is an instantaneous quantum leap" (1991, 20), a sudden change in perception. This cycle of interiorization, condensation and reification is repeated generating a hierarchy of more and more abstract concepts.

What is the importance of achieving this duality of conception? Sfard argues that while the operational is necessary and also sufficient for certain lower order mathematical thinking, the structural is necessary to manage more complex processing. Operational thinking involves sequential processing; compacting processes into mental objects protects the "working memory against overflow" (1991, 29). New knowledge can be assimilated effectively into hierarchical structures, enabling meaning and allowing faster recall (1991, 28). Without developing deeper structures, schema may become unwieldy sequences of actions unsuitable for further growth. In problem solving, the higher structural level is used to get a 'general view'. Mathematical activities require an intricate interplay between operational and structural versions of the same idea.

The problem with achieving reification.

Reification occurs as a flash of insight and Sfard suggests it is not clear what triggers it. However, Sfard argues multiple representations are significant in achieving this qualitative leap, such as naming objects, using symbols and picturing graphs. Furthermore “reification of a given process occurs simultaneously with the interiorization of higher-level processes” (1991, 31). In effect, to assimilate the newly forming object it is necessary to trial some actions on it; but if the object is not yet formed how can an action be carried out on it? Sfard calls this the “vicious circle” of reification, and draws out from this the implication that proficiency and comprehension need to go hand in hand. The challenge is to provide an appropriate mix of experiences for the individual to optimise their opportunities for achieving the sudden insight that brings reification.

Reflections from use in the classroom.

Due to limited time to experiment, I have a restricted set of examples from the classroom; however some interesting points emerge. The first two examples are from first year A-level classes, studying AS level mathematics.

I used the idea of approximating area under the graph with the students using pencil and paper to draw a quadratic and approximating the area by a small number of trapezia. I then asked them how could they improve the approximation and groups fairly quickly came up with the idea of increasing the number of strips and making the strips narrower. After one further drawing I then showed them the results in Autograph on the screen. I then increased the number of strips and asked them to predict what the value for the area looked as though it would approach. Through use of Autograph it was fairly simple and quick to show increasing number of strips with rapid feedback as to the value of the new area. Students were comfortable that the area was approaching some number and that in theory we could continue to make the strips narrower and narrower.

In another lesson with a different group, having been through the above activity previously, I introduced the concept of area-so-far, based on ideas explained in Tall (2003, 16 - 18), and reminded them of the concepts of rate of change and that we had named the process of anti-differentiation integration. I then went through the argument of local flatness on a whiteboard and proceeded to the result that the rate of change of area is the graph function. This was immediately greeted by one student with the comment that he understood what was going on, whereas at other times he got lost – “when you explain it like this I understand it”. He transmitted a feeling of illumination and a recognition that the style of explanation was different.

Another lesson illustrated that by using a concrete and iconic approach combined with a process, a student achieved a better understanding. Using an idea from Watson et al (Watson, Spyrou and Tall 2003) I reviewed two-dimensional vectors with a second year A-level group (A2) before proceeding to three dimensions. I asked them, in groups, to draw a picture of a hand on a large piece of paper and then another, translated. I then asked them to draw a line representing the motion; and then another; and then another (see Mason, 2005, page 2, for this technique to achieve generalisation), and then challenged them to draw a line which did not touch the outlines. The intention was that by focussing on a number of equivalent physical movements, the dynamic processes involving equivalent free vectors would be emphasised rather than the static notation of a vector as an object. But also by forcing a change of attention away from the hand to finding a line not touching it this was also pressing the students towards reification. While one student said this was going back to primary school another student who over the year had become increasingly disillusioned with the mathematics said - “I'm having a good maths day today, I understand what I'm doing”.

Evaluation and implications for teaching

In returning to teaching after many years out of the profession, I was immediately struck by the difficulty that AS students had with manipulation of simple algebra, despite the fact that fifty percent of them had achieved grade A at GCSE. My concern was that if remediation was not achieved soon then by the time we got to the harder parts of the AS course they would be lost in a mass of meaningless symbols and rules for their manipulation.

The theory proposed by Sfard provides good model to analyse the students' learning. My students have learnt a number of rules which for the most part have limited meaning for them but which they have reasonably successfully applied in examinations. Consequently that is what they see mathematics as consisting of and expect more of the same at AS level. They have no doubt previously performed practical work but (1) see that as elementary and something they have left behind and (2) they have successfully worked at the operational level but have not made the jump to the structural level for many of the concepts.

Up to this time my teaching for the most part had consisted in working at the symbolic level, in Bruner’s terms, while in the majority of cases the students are thinking at the enactive or iconic level. Consequently they struggled to link the course to what they already knew, lack the means to achieve meaning with the new concepts and continued with their strategy of learning rules and trying to produce the right answer through appropriate manipulation. While this strategy may have worked at GCSE, having to advance to one or more levels of abstraction further becomes in many cases a jump too far.

Where I have introduced practical work and processes and linked it to the concept that encapsulates it I have had positive feedback from a few; feedback of a nature that it is clear that the students concerned have experienced illumination.

The two missing elements for many students are experience of process and, critically, the reification of the processes into a mental object. The use of software can greatly help here. It easily generates processes with varying parameters and provides the dynamic imagery to help achieve reification. The dynamic nature of graphing software such as Autograph or the manipulative facility of Cabri allow processes to be carried out which help bring the student to the edge of reification.

The dynamic nature of certain software is a key element as it allows the mind to focus on critical features whereas in a static image there is lots of information and it is not necessarily evident as to what to focus on. Further, it allows the mind to see what changes, what stays the same, what is connected – fundamental mathematical ideas of invariance and covariance.

Non-ICT activities can also help here, as in drawing a couple of graphs or drawing translations on paper. ICT can be used as a complementary tool to then generate many examples quickly, to provide dynamic images that focus the mind on the critical features, and that provide rapid feedback.

As concepts become more abstract, so ICT can help where other tools may not be of use. Sfard describes reification as a process that happens repeatedly, mental objects building one from another. Goldenberg argues that while at the lower levels physical objects can be used for enactive activity, but as the concepts become more abstract "computers can provide interactive 'virtual manipulatives' where physical devices do not exist" (2000, 1).

While it appears to be an open question as to what triggers reification, Sfard argues that multiple representations are very helpful (1991, 20). This is also identified by Bruner (1966, 65 - 68). Therefore use of multiple representations would be recommended in using software.

While the demonstration of processes to a whole class is very useful, Bruner's theory suggests that students will benefit from the hands-on activity themselves, to perform the enactive mode. Furthermore, use of keyboard and mouse to represent graphs is one step removed from the actual hands-on pencil and paper activity, an iconic activity rather than enactive. It follows that pencil and paper activities are still critical. However, use of technology such as distance monitors whereby the student recreates a graph by walking with the sensor would also be an enactive activity. Hands-on use of dynamic software could also be addressed by using graphic calculators.

A number of sources have pointed out that language is important in concept development; in particular Bruner identified its role in assisting symbolic thought. It would make sense that by making investigations of dynamic images in groups that students would have the opportunity to use language to describe what they see.

Students' attitudes and abilities with the ICT facilities are also factors. I have found in the classroom some are keen to have hands-on experience, others are reluctant. Furthermore the students need to be comfortable with the ICT tools provided, having had the appropriate training and experience of using them.

The teacher also needs to have time to become familiar with the tools to create and deliver the lesson (Goldenberg 2000, 8). Moreover time is needed to analyse the mathematical ideas to be taught, identify the concepts and processes they consist of, and time to develop the supporting material, using ICT appropriately. Indicative of the Ofsted comments on the diminishing role of ICT, the Edexcel textbooks for the A-level modules Core 1 to 4 (Edexcel 2004) make no mention of ICT and tend to approach topics in the symbolic mode. In contrast SMP books produced thirteen years earlier take a concrete and iconic approach, use graphic calculators and computers, and provide datasheets to help with setting up software examples (SMP 1991, v and viii).

Conclusion

Connecting the ideas of process and object, operational and structural thought, to the facility of using dynamic imagery, and providing multiple representations, is potentially a powerful approach. This allows students to focus on what is key for the development of concepts, seeing patterns and making connections, giving them the opportunity to achieve the flash of insight that is reification. In this way perhaps students will come to appreciate that mathematics does have meaning, a meaning that is constantly evolving for each individual as the process of exploration continues:

> "We teach a subject …..to get a student to think mathematically for [her/]himself, ….. to take part in the process of knowledge-getting. Knowledge is a process not a product" (Bruner 1966, 72).

Note

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