Visualisation of cosets and its impact on student engagement with Group Theory

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Group Theory is considered by students as one of the most taxing topics of their university studies. In this study, I scrutinize the role of visualisation in the student engagement with Group Theory, focusing, in particular, on the notion of coset which its introduction, according to the data, is the first major milestone students need to face and overcome. The data consists of: observation notes and audio-recordings of lectures and group seminars of a Year 2 course; student and lecturer interviews; and, coursework and exam papers. Moreover, I offer some excerpts from the collected data which demonstrate the link between the ability to visualise the concept under study and the development of student engagement.

Keywords: undergraduate mathematics education, student engagement, Goldin's theory of affect, visualisation, coset, Group Theory

Introduction

Group Theory is one of the mandatory courses taught usually in the second year of a Bachelor degree in Mathematics and, more often than not, it is considered by the students as one of the most demanding ones. Often students tend to avoid third-year or further courses in this area of mathematics, after their first learning experience with it. Previous research (e.g. Nardi, 2001) attributes student difficulty with Abstract Algebra in general, and Group Theory in particular, to its multi-level abstraction and the less-than-obvious, to students, *raison d'être* of its concepts. Furthermore Abstract Algebra is amongst the first courses in which students are not able to cope with by just memorising formulas or by "just learning 'imitative behavior patterns'" (Dubinsky et al, 1994, p268). Also, as it is often suggested by research (e.g. Millet, 2001), several teaching techniques such as lecturing to large student audiences has a debatable effect on student engagement, especially in a topic such as this, where interaction with the teacher is even more necessary for novice students.

This study's major aims are to examine closely mathematics undergraduates' learning experience in Group Theory and in particular the effects on the student engagement after the introduction of the notion of coset. I am particularly interested in the link of cognitive and socio-affective aspects of the students' experience since, as Goldin (2000) proposed, affect is "critical to the structure of competencies accounting for success or failure" (p211). Below I present some observations on the students' apparently diminishing engagement, after the notion of coset is introduced. The excerpts were extracted from the interviews with the students, the lecturer and the seminar leaders and assistants, in which they have discussed with the author, among other issues, about how students visualise abstract concepts and the effect on their learning experience. Moreover, and given that constructing appropriate visual imagery is often described in research as providing crucial support to this meaning-bestowing process (Zazkis et al, 1996), I highlight how their relationship with Group

Theory is affected by the difficulty to visualise, giving special emphasis on the concept of coset.

Methodology and context of study

This study is a scion of my doctorate research which aims to examine several aspects of the Year 2 mathematics undergraduates' encounter with Abstract Algebra. The data was extracted from a course currently taught in a well-regarded mathematics department in the UK. Data collection took place in the Spring Semester of a recent academic year. The course was mandatory and 78 students attended it. It was spread over 10 weeks and there were 20 hourly lectures. Additionally to the lectures, there were 3 cycles of seminars, in Weeks 3, 6 and 10. The role of the seminars was mainly to support the students with their coursework. There were 4 seminar groups, with about 20 students in each. In each seminar group there was a seminar leader, a full-time faculty member of the department, and a seminar assistant who was a PhD student. All members of the teaching team had related research interests. The students submitted the coursework at the end of the semester. This was marked and returned to them soon after.

In the lectures there was no significant interaction between the lecturer and the students. The lecturer, a very experienced mathematician, was writing extensively on the chalkboard and was commenting orally alongside. In the seminars the students were expected to work on problem sheets, distributed to them earlier in the preceding weeks, and arrive having prepared questions. They had the opportunity to ask the seminar leaders and assistants anything they had difficulty with and receive help. The lecturer was also available during 'office hours' for the same purpose.

Data was collected by the author and consists of:

- **Lecture observation notes**. These covered: record of student attendance; instances of interaction between the students and the lecturer; verbal, body or other evidence of student (dis)engagement and emotional response to the lecture; and, general observations of lecturer and student behaviour.
- **Lecturer notes**: notes of what the lecturer was writing on the blackboard.
- **Audio-recordings** of the 20 **lectures**.
- **Audio-recordings** of 24 **seminars** (2 recorders in each of the 12 seminars; one on the seminar leader and one on the seminar assistant) in which I have captured all conversations with students during which they predominantly discuss difficulties with certain items in the problem sheets.
- **Interviews with** 13 out of the 78 **students** who made themselves available on a voluntary basis, the 4 **seminar leaders and assistants and the lecturer**. There were three cycles of interviews, at the beginning, the middle and the end of the course, in which students discussed their learning experience in Abstract Algebra. The discussions with the lecturer, who was also one of the seminar leaders, covered learning and teaching issues as well as institutional and administrative issues. Interviews with the seminar leaders and assistants were mostly about their discussions with the students during the seminars, and their general views on pedagogical issues.
- **Student coursework**. Students were given three problem sheets in Weeks 2, 5 and 9. They had to work on these before the seminars on Weeks 3, 6 and 10. They had to work on all problems, but they had to hand in only a selection of these in Week 12. The selection of problems to be assessed was announced to them after each seminar.
- **Marker** (seminar assistant) **comments on student coursework**.

- **Student examination scripts** collected at the end of the academic year.

Data Analysis and Theoretical Framework

The decision to investigate further the issue of engagement with relation to visualisation, and in particular, with the ability to visualise the concept of coset is based on two facts. Firstly, on the existing literature which suggests that the students' relationship with visualisation is problematic (e.g. Presmeg, 2006) and secondly, on the fact that the majority of students were expressing their necessity to visualise concepts in order to 'understand' the course, given the additional element of abstraction of Group Theory which was an extra obstacle.

Scrutinizing the lecture data (audio recordings and notes), amongst the many themes that emerged (Ioannou & Nardi, 2009b) were: the diminishing student engagement over the ten weeks of the course; and, the variable responses to uses of visualisation in the lectures. Evidence on the former (lecture observation notes and the student and lecturer interview transcripts) suggested what has been listed in (Ioannou & Nardi, 2010) as: a *pathology of absence* (decreasing student attendance); a *pathology of presence* (increasing disengagement of those still attending); and, *explicit student expression of emotion* (statements of disengagement increasing gradually in frequency and power over the three cycles of interviews). Furthermore, choosing to focus on the role of the concept of coset is based on the fact that the aforementioned phenomena amplified in frequency and potency when the notion was introduced.

For the purposes of this study, I am using Goldin's theory of affect (Goldin, 2000) according to which affect in mathematics is described in terms of four elements: beliefs and belief structures; attitudes; emotional states; and, values, ethics and morals. The concept of "local affect" defined as "the rapidly changing states of feeling that occur during problem solving – emotional states, with their nuances" (Goldin, 2000, p210), is particularly important. Goldin describes eight such emotional states and several possible ways in which these affective states may lead to certain problem-solving strategies. However, there may be a possibility Goldin's model to be expanded in wider range of students' learning experiences, not just strictly problem-solving related ones.

According to Goldin (2000) at the first stage of problem solving students may experience feelings such as **curiosity**, **puzzlement** or **bewilderment**. Following this first stage there are two possible *affective pathways* (Weber, 2008, p82): *favourable* (i.e. emotions of **encouragement**, **pleasure**, **elation** and **satisfaction**) and *unfavourable* (i.e. emotions of **frustration**, **anxiety** and **fear/despair**). Moreover, "these pathways in *local* affect lead to *global* affective structures such as specific representational schemata, general self-concept structures as well as (particularly the second pathway) self/mathematics/science/technology resentment." (Ioannou & Nardi, 2009b, p38)

According to Weber (2008) these affective pathways may be *self-strengthening* if their duration is long. A repeated emotional experience is possible to cause stable attitudes and beliefs that may be related to particular cognitive responses (Goldin, 2000). Moreover, "mathematical understanding is organic, since, when students feel that they have achieved some understanding in one mathematical topic and consequently they find it pleasurable, they want to extend their understanding with regard to this and other mathematical topics." (Ioannou & Nardi, 2009b, p38)

Visualisation and Engagement

Scrutinizing the 39 student interviews and the 16 staff interviews, there have been found 20 statements of varying length about the notion of coset and how this influences the students' learning experience. In what follows, I present a sample from this set of statements and additionally present examples of images produced by the lecturer and two students.

In a question to a student what a coset is and if he/she can produce a mental image, although the introduction of the definition, escorted by the relevant illustration, was recent, the student replied:

> "S: A coset... I don't really know... I don't – I'm not too sure in my head at the moment, about a coset, it's the thing I struggle most with, like, with um... it's cosets, when I – I don't think I can picture them, and I think – cos obviously I'm a visual learner, I learn – I'm better if I can picture in my head, but a coset, at the moment, I'm just... a bit..." (LH2)

In the above statement one can notice the desire of the student to have a mental image of cosets and the on the other hand, the difficulty that faces to do so, resulting frustration and anxiety. Another student when asked the same question replied:

> "I: Um... do you have a mental picture of a coset?
>
> S: Coset... mm...No, I can't, I can't really think of anything visually, for that..." (MH2)

Student's failure to reproduce the already given image of a coset in the lecture notes, reinforces the claim of Ioannou & Nardi (2010, p9) that "the images given by the lecturer do not necessarily have the intended impact since they are rarely noticed or adopted by the students".

Students' reluctance to reproduce the images given in the lecture notes was confirmed by the fact that only 2 of the 13 students gave a substantial, but not always clear, image of coset. Figure 1 presents the image given by the lecturer and Figure 2 the reproduced images by the 2 students during the interviews.

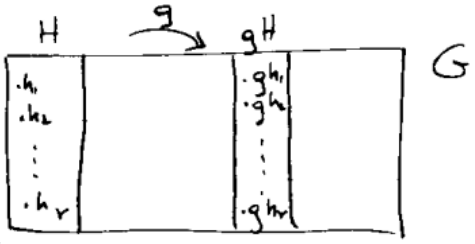

Figure 3. Lecturer's illustrations of coset

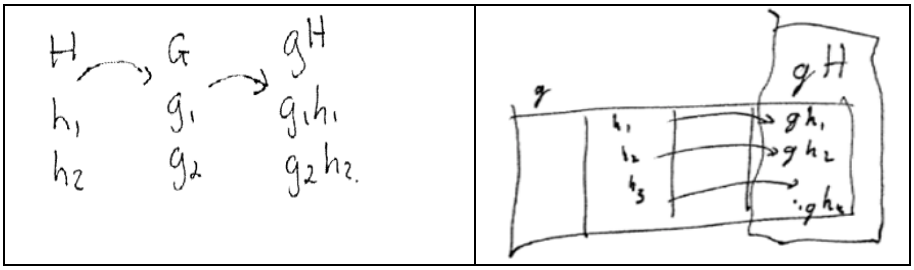

Figure 4. Students' illustrations of coset

In the following statement it is apparent that the first stage of learning experience with Group Theory develops towards an unfavorable affective pathway.

The initial curiosity caused by the introduction of a new course is developing into anxiety.

> "S: The very first part of the course with the groups and the combinations was ok, but then it became more difficult, especially with the first coursework…
>
> I: I guess you mean cosets and factor groups…
>
> S: Yes, yes… these concepts were much more complicated and they were difficult to understand…" (FP2)

As part of a discussion about an exercise that involved cosets, a student describes in the following statement his/her current feelings about the course, now that cosets have been introduced.

> "Yeah, and like... in the – in the proofs as well, it's like – oh, but the – that means this, and it's just trying to understand just why that means that, and because I can't see it in my head, and I can't visualise it, it just – I don't see why they're so you know – like it goes and therefore this, and I'm just like – but why?" (KL2)

The initial puzzlement that may this student have felt at the beginning of the course has developed into intensive bewilderment and despair, which is quite apparent by noticing not just the contents of his/her utterance but also its structure.

It is interesting to add that experienced mathematicians, who taught in the three seminar cycles, agree with the claim that introduction of cosets is an important milestone in the learning experience of novice students in their first encounter with Group Theory. The following statements substantiate this claim.

> "Cosets, I think are a particularly formidable barrier..." (GE2)
>
> "...when these first came up cosets – what does it actually mean - it is a hard concept to get your head round. Especially because of the fact you have two cosets and they kind of lay more differently but the same cosets – they have got this thing that show these two cosets are the same these two elements have the same action so the idea that you can lay all these different representatives but they are not - you need representatives it is a difficult thing to understand." (SA2)

Conclusion and further steps

Even though students emphatically express their need for mental images that will assist them to grasp the abstract concepts of Group Theory and especially the notion of coset, they are at the same time reluctant to attempt a construction of such images or engage with the images on offer by their lecturer. Additionally, students do not use the images on offer as explanatory tools that will allow them understand better the related definitions. Furthermore, ability to visualise depends on the level of abstraction of the course under study. (Ioannou & Nardi, 2010)

The abstract nature of cosets makes this concept one of the most difficult for the novice students to grasp and visualise. Considering also the fact that cosets are a prerequisite for the definition of other abstract concepts, such as the quotient structures in Group Theory, it reinforces their significance in the learning process.

In Goldin's terms the emotional states recorded in the above quotations may progress into longer-lasting 'unfavourable' emotional structures. Further research will aim to trace these structures, and their impact on students' competence in Group Theory, across all other data sources (e.g. coursework and exam papers). In addition, special emphasis will be given to the concept of coset and its impact on the student engagement with Group Theory. Furthermore, the research will aim to examine the

role of coset in more detail, from three different perspectives: the emotional-affective, the cognitive and the teaching perspective.

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