A Student's Symbolic and Hesitant Understanding of Introductory Calculus

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In this paper I discuss a study that looked at one student's understanding of calculus, and used the framework of Tall's theory of Three Worlds of Mathematics to determine the embodied and symbolic nature of that understanding. Initially, the student's understanding of calculus was explored through a task interview using calculus questions designed to elicit embodied and/or symbolic understanding. Results showed that this student predominantly demonstrated a symbolic understanding, with a very limited embodied understanding on the particular tasks. It was also during this interview that the student exhibited the phenomenon of searching for reassurance as to whether he was answering the task and interview questions correctly. This paper discusses this search for reassurance, speculates on potential causes, and argues that there may be a relationship between this search for reassurance and the student's symbolic understanding of calculus.

Keywords: reassurance, calculus, symbolic understanding, didactic contract, institutional norms, three worlds of mathematics

Introduction

The research presented in this paper was part of a study that sought to explore the experience of a first year undergraduate calculus student and how that student's experience in calculus may have contributed to his symbolic (algebraic procedural) understanding and/or his embodied (physical or geometric) understanding, of calculus, and particularly the connections between these understandings. This initial question was examined with regard to a portion of a particular philosophical theory, namely Tall's three worlds of mathematics. The data for this study were collected from a clinical task and reflection interview with one university student who had recently completed a first year calculus course. The purpose of the interview was to examine the understanding the student demonstrated on calculus tasks and how the student spoke about his understanding and experience in introductory calculus. During the clinical interview, an unexpected phenomenon appeared: the student's need for reassurance that he was correct. This was surprising, as the student in this study had completed introductory calculus with over a 90% average and thus, it was assumed that the student was confident with calculus. This paper will begin by briefly describing the study as well as summarise the results. I will then discuss the student's need for reassurance that appeared during the clinical interview. In the conclusions of this paper, I will hypothesise some reasons why the student in this study, as well as other students in general, may feel the need to look for reassurance about how they are doing their mathematics.

Background

I developed the research problem of this study from a combination of my own experience as a mathematics student, tutor and instructor as well as an examination of Tall's philosophical theory of three worlds of mathematics (Tall 2003, 2004a, b). This philosophical theory suggests that all types of mathematics can be divided into one of three domains, or worlds: the Conceptual-Embodied world, the Symbolic-Proceptual world, or the Formal-Axiomatic world. The Conceptual-Embodied world involves a combination of Bruner's Enactive and Iconic modes of representation (Bruner 1966; Tall 2003). According to Bruner, the Enactive mode of representation signifies an action or actions while the Iconic mode of representation is attributed to all visual or image organisation. Thus, Tall's Conceptual-Embodied world of mathematics is a combination of actions on organised visual or image representations of the mathematical world (Tall 2003). The Symbolic-Proceptual world is a combination of processes and concepts acting together, or procepts, which utilise the formal symbolic systems such as arithmetic, algebra, or, in the case of this study, symbolic calculus. The Formal-Axiomatic world uses logical deductions, formal definitions, and axioms to construct axiomatic mathematical systems, such as analysis or field theory. For the purposes of this paper, I will refer to the Conceptual-Embodied world as the embodied and the Symbolic-Proceptual world as the symbolic for the sake of brevity.

Tall's theory has been applied to many areas of mathematics. For example, one study used Tall's theory to investigate university students' embodied and symbolic understanding of a specific concept within linear algebra (Stewart and Thomas 2006). Similar to my own investigation, these researchers focused on individual student's understanding within the explicit worlds. Their findings conclude that developing an understanding of the connections between the embodied and symbolic world can contribute to overall success. Nogueira de Lima and Tall (2006) used the theory of the three worlds to try to explain 15-16 year old students' misunderstandings and difficulties with learning the concept of equations. Similar to the previously mentioned study, Nogueira de Lima and Tall also concluded that developing a connection between the embodied world and the symbolic world can contribute to a better understanding of the concepts.

For my study, Tall's theory, and particularly his descriptions of the different worlds of mathematics, was applied to the participant's answers on introductory calculus tasks. This was intended to analyse the different worlds the student's answers could be categorised in to see if there was a cross over between worlds in the questions. From my experience, a traditional first year undergraduate calculus class focuses on the graphical ideas of rate of change and cumulative growth and the use of the rules of differentiation and integration for symbolic manipulation. This was also the case for the course the student in the study took. Thus, I made the assumption, which Tall's theory supports, that there was the potential for a different understanding of the embodied and symbolic worlds in calculus. I then wanted to explore how the experiences of a student could potentially contribute to the different understandings. That is, if a student experienced mathematics as either mostly from the symbolic world or from the embodied, or a combination of the two, would this correlate to the understanding the student developed? The Formal-Axiomatic world was not included in the study since it was not a significant part of the introductory calculus course that the participant had taken.

The Problem

For this study I wanted to know if there was something about an individual's experience that contributed to his or her developed understanding of the mathematics, as categorised by Tall's worlds of mathematics (Tall 2003, 2004a, b). I chose to position this study within the context of a first year undergraduate calculus class[i1] since calculus is required by numerous undergraduate programs, and is one of the most commonly taught mathematics courses at university. For example, at one university in Canada, undergraduate programs in Mathematics, Physics, Commerce/Business, Economics, Engineering, Health Science, General Science, and Environmental Science all require students to complete a course in calculus. Thus, because of the volume of students who are required to take calculus, studying how students' experience contributes to how they develop their understanding could be informative to educators. Thus, the research question for this study was: 'How does a particular student's experience in a calculus course contribute to his development of a symbolic and/or embodied understanding of the mathematics that was taught?'

I chose to begin my investigation by looking at only one student's experience, as this was a pilot study to test the interview task questions. My participant for this study was not chosen at random as I knew Andre prior to the study. Andre had hired me as a private tutor to help him through some pre-university mathematics classes. Despite this connection, I had no contact with Andre in any capacity while he was completing his university calculus course and felt that our prior connections would not bias the data.

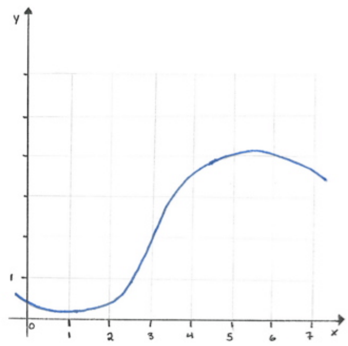
I designed the initial research study after the clinical interview techniques for mathematical thinking outlined by Ginsburg (1981). Ginsburg recommends three clinical interview techniques for mathematics education research. These include a discovery technique, which is aimed at discovering the cognitive processes actually used by the individual in a variety of different contexts, an identification technique, which is aimed at identifying the describing the intellectual phenomena that are discovered, and a competence technique that is aimed to establish competence, not just performance. For this study, my interview with Andre was a combination of Ginsburg's discovery and identification mathematical clinical interview techniques. I designed 10 open-ended mathematical tasks to be completed by Andre during our interview for the discovery portion (partially reproduced in Figure 1). These tasks were designed so that they could be answered both in a symbolic and/or embodied way, in an attempt to try to comprehend the understanding that related to the processes that Andre was using. A professor in Mathematics Education with calculus experience reviewed and agreed with me about the design of the tasks. The interview also consisted of a series of semi-structured questions about Andre's calculus class and his previous experiences in mathematics.

Once Andre had attempted all of the tasks, I engaged him in a reflection about the tasks. I prompted him to speak about how his previous experiences in mathematics, and specifically in calculus class, contributed to how he completed the tasks. At all points when I was with Andre, I made written notes of comments Andre made that I deemed important. The semi-structured reflection interview was audio-recorded. Further to my field notes and the audio recording, I collected the paper that Andre used when solving the mathematics tasks in order to have a written record of his problem-solving steps. After the interview was completed, my written notes and

[1] Henceforth, when calculus is mentioned, it refers to first year introductory undergraduate calculus, unless otherwise stated.

the interview transcripts were thoroughly read, analysed, and coded according to themes that emerged. I initially analysed the data for signs of Andre's embodied or symbolic understanding of the calculus concepts in terms of Tall's descriptions of these worlds (2003, 2004a, b). For example, I looked for physical drawings or graphs to demonstrate an embodied understanding and I looked for symbolic manipulation and algorithms for a symbolic understanding, to name a few, as well as any combination of these and other processes Andre may have used. I then examined how this related to how he spoke about his experience.

2) For the function *f* (whose graph shown) arrange the following numbers in increasing order and explain your answer: 0, 1, $f'(2)$, $f'(3)$, $f'(5)$, $f''(5)$



4) At what point on the ellipse $x^2 + 2y^2 = 1$ does the tangent line have a slope of 1? Show your work and explain your answer.
5) What is meant by "rate of change" in terms of calculus?
9) If $w'(t)$ is the rate of growth of a child in pounds per year, what does $\int_5^{10} w'(t)dt$ represent? Explain your answer.
10) If $f(x)$ is the slope of a trial at a distance of *x* miles from the start of a trail, what does $\int_3^5 f(x)dx$ represent? Explain your answer.

Figure 1: Example of Calculus tasks

Results

To summarise, the results showed that Andre demonstrated a very strong symbolic understanding of calculus but not a solid embodied understanding. He had a dominant tendency to want a procedure when solving different types of calculus tasks and avoided the use of graphical representations to help in the solving process.

A good example of Andre's symbolic understanding of calculus occurred when he was trying to complete the second question of the tasks (Figure 1). This question showed a randomly drawn curve. Andre began the task by first wanting to identify the function equation of the curve and decided that the function was actually f(x) = sin(x). When I asked Andre why he first wanted to make the curve a sine curve he reported it was: "Because I am probably used to calculating the derivatives and integrals, so I was thinking that what was the derivative of a sine function or cosine function and I tried to see what it would look like..." This response shows that Andre was inclined to gravitate towards what he described as calculating when it came to calculus questions.

The desire to 'calculate' in the questions continued throughout the entire time Andre completed the tasks. When questioned about why he wanted to find procedures or calculations for the tasks, Andre said he was not taught any theory about the mathematics, but that he was only given the formulas and then used them to do example after example. He also said that the examples were very similar to what would then be on the tests, so that's what he learned. Thus, the way in which Andre described his experience in calculus showed an emphasis in the teaching on what Tall would categorise as a symbolic understanding of mathematics (Tall 2003, 2004a, b).

Another example that suggests Andre's strong symbolic understanding had to do with sketching the curve of functions. When I asked Andre what he found easiest in calculus, he stated curve sketching and said that was because "you are given an easy function to trace, you follow your steps...". Andre then explained to me the procedural steps he would follow to sketch a curve. During the explanation, Andre mentioned using a grid method to record the answers. Interestingly, this grid method has nothing to do with the mathematics, but could be described as a socio-mathematical norm (Yackel and Cobb 1996). Thus, from his initial comment and further explanation, including the mention of the grid, I concluded that to Andre, 'easy' means a step-by-step process to follow, or procedures, which usually take place in the symbolic world of mathematics.

Despite Andre's overwhelming tendency to respond to calculus from a symbolic understanding, he did have a selected understanding of calculus as it related to the embodied world. For example, on question 2, once it was clarified for Andre that he was looking at an arbitrary curve, he was able to solve the question by visually representing the different slopes. When I asked him what he looked for when solving this question he said "slopes" and commented on placing them in order according to how much they increased in steepness. Likewise, with question 5, when I probed Andre to explain what he meant by variation in his definition of rate of change he said he meant that he would look between two x values and see how much y has changed in that period. Although this is not a perfectly clear answer, it shows that Andre is still thinking about rate of change or slope as something he can physically see as well as calculate. Overall, from the interview, I concluded that the understanding Andre demonstrated was related to his experience in how mathematics was taught to him. Andre stated that calculus was taught focusing on the procedures and practicing examples using the procedures. Taking this into consideration and the fact that procedures in introductory calculus tend to involve the symbolic world, it is not surprising that symbolic procedures are mostly what Andre learned from the class.

Need for Reassurance within a Symbolic Understanding

It was during the analysis that another theme appeared. This was Andre's tendency to show a need for reassurance that he was doing the tasks correctly or answering questions correctly, such as hesitating on questions to ask me if he was answering correctly. In this section I will summarise my findings in conjunction with the theoretical framework of Sierpinska's that focuses on a similar theme, namely the student's need for the teacher to tell them if they are correct (2007). Throughout the summary, I will demonstrate how Andre's responses can be situated inside this framework and suggest that this need for reassurance can also be linked to Andre's symbolic understanding.

Sierpinska's findings were similar to my own in that the participants of her study demonstrated and expressed a need for reassurance from their instructor (2007). Sierpinska speculates that this need for reassurance, or for the teacher to tell the student if they are correct, could be attributed to epistemological, cognitive, affective, didactic, and/or institutional reasons (2007). Reviewing Andre's responses, I would categorise his need for reassurance into specifically institutional and didactic reasons, but there could most likely be affective reasons as well.

Andre demonstrated a need for reassurance, which could be attributed to institutional rules and norms, while solving task question #4 (Figure 1). While attempting this question Andre stopped midway through, drew a smiling face, and went on to the next question. This was surprising since Andre was actually solving the problem in a correct manner. When questioned as to why he stopped, Andre said, "Because I wasn't really sure…since we didn't have the word implicit *(referring to implicit differentiation)* I didn't know if I was supposed to do that." This speaks to the convention Andre was taught in class – to only solve a particular way if primed by a particular word. Since that word was not present, Andre was not sure about his action and thus stopped trying to solve the task question altogether. This example speaks to his need of reassurance not just from the instructor, but from his prior experience with calculus questions. In other words, Andre had experienced reassurance when taking calculus in that when he saw the words 'implicit differentiation', then he felt reassured that he was supposed to solve the question in a particular way. Andre's experience of needing a priming word in the question is an example of an institutional norm that somehow became viewed by Andre as necessary for mathematical validity. This example also gives support to the results discussed earlier that Andre has a strong symbolic understanding of calculus. With a symbolic understanding, Andre is comfortable with questions that require step-by-step procedures for solving. Andre had most like developed his understanding in the way Siepinska described, as "justified on the basis of their acceptability by the school authorities, not on their grounding in an explicit mathematical theory" (2007 16). Andre demonstrated here that one of the steps he required for solving question like #4, which was based on his experience with acceptable school authorities, was to see the word 'implicit' written within the question. Sierpinska suggests the reassurance phenomenon could exist because of the institutional rules and norms that students are taught, which equate to mathematical validity. This reassurance can also be attributed to some of the procedures used in solving calculus problems. In class, a particular step-by-step procedure might be taught as the only right way to solve a problem, even if it is only conventions set by the instructor.

Andre also demonstrated a need for reassurance that could be linked to the didactic perspective. When I asked Andre why he had solved task question #9 a certain way and particularly why he sketched the area under the curve, which happened to be a correct way for solving the question, Andre got a little upset and stated "Umm, because that's what I think it was...Okay, I'm a little bit confused." This example shows that when I asked Andre to justify why he solved something in a particular way, he claimed to become confused. I speculate that he was not necessarily confused but started to question his answer and lose his confidence when I asked him to explain what he had done. On further probing it became clear that he thought I was only asking him because he had not solved the question correctly, which was not the case. From the didactic perspective, Sierpinska suggests that in mathematics, the teacher gives the task and it is the student's job to produce an answer. According to this model, Sierpinska claims, "the teacher is assumed to know the correct answer" (2007 15). There is also the suggestion by Sierpinska that there are many tasks in school mathematics where it may be impossible or difficult for the student to verify the answer. This example demonstrates Andre's understanding of the didactic perspective between instructor and student, that is, that the instructor holds the answers. This example also makes me wonder about the understanding Andre has of the role of questioning in mathematics. Andre lacked the confidence to explain and argue for his method of solving this particular task. Instead, he needed to be reassured that questioning how and why he solved a task in a certain way was not an indication that he had done anything incorrectly. This lack of confidence again may be linked to Andre's symbolic understanding. Since Andre had learned the mathematics as step-by-step symbolic procedures, he lacked the knowledge of why he was doing particular steps and only knew that it was part of the process. I questioned Andre as to where he learned to draw a curve and shade the area underneath to represent integration. He said it was something the teacher always did before he solved the question. Thus, Andre was mimicking a process he experience in class without understanding why it was significant.

There were also numerous times throughout the interview when we were discussing what he had done when Andre would pause during an explanation and ask "Right?", "Correct?", or "Do you know what I mean?" The frequency of these pauses for reassurance was such that it was very noticeable and made me conclude that Andre was either not confident in what he was doing or expected me to reassure him throughout the interview. As was mentioned earlier, Andre had received the highest grade possible in his calculus course and thus, I assumed he would have developed some level of confidence when attempting calculus questions. These verbal pauses for reassurance make me wonder if Andre completed his calculus course with a lack of confidence. Conversely, it may be the case that Andre's didactic expectations were for me to confirm this processes while he was solving.

According to Sierpinska, one of the affective reasons for a student's need for reassurance may come from the certain words that are used in mathematical discourse. In particular, according to Sierpinska, mathematical discourse uses the terms 'right' and 'wrong' instead of 'true' and 'false', the latter of which are more appropriate in mathematics. Sierpinska suggests that the words 'right' and 'wrong' are "emotionally laden, especially when uttered in relation with a student's work" (2007 13). Thus, students desire to hear that their work is 'right', as that has positive emotional connotations as well as connects to the notion that there is one right way to do things, which is a symbolic/procedural view.

Although I did not ask Andre directly how he emotionally felt about the questions, I did notice and recorded a lot of apprehension when he was not sure if he was correct. Likewise, one of the reasons Andre may have stopped solving task question #4 may have been that he did not want to lose morale in the case he was proceeding incorrectly, as well as the lack of the priming words 'implicit differentiation'. This may also have been why Andre continued to ask me for reassurance that he was answering the questions correctly. Similar to the other potential reasons already mentioned, Andre's emotional need for reassurance may also be supported by his symbolic understanding. Without a strong embodied understanding of the mathematics, which provides the knowledge that allows students to develop an understanding of the justification of the procedures they take, students may look for positive reinforcement that they are proceeding correctly as they have no other indicators available to them.

Conclusion

Overall, it was unforeseen that Andre would exhibit throughout the interview a need for reassurance that he was answering the mathematics task questions and/or the interview questions correctly. Now, after examining the need for reassurance in relation to Andre's strong symbolic understanding, there seems to be strong links between the two. More specifically, it seems that the symbolic understanding, based on step-by-step procedures for solving mathematics problems, may also lead to students' feeling hesitant about mathematics.

I believe it would be beneficial to further explore this phenomenon of a need for reassurance in mathematics and, in particular, if this phenomenon occurs more often in individuals who demonstrate strong symbolic understandings of mathematics. At the same time I had at one point been Andre's tutor and part of me wonders if Andre looking for reassurance from me comes from the fact that he still sees me as a teacher when we are discussing mathematics. This might play a role, but I believe that it has more to do with a combination of the reasons discussed in Sierpinska's (2007) research as well as his developed symbolic understanding of calculus. Thus, my concluding hypothesis is that there is a link between a student's symbolic understanding and demonstrating a need for reassurance. In future research, I would like to investigate on a larger scale whether it is mostly students who demonstrate a symbolic understanding that search for reassurance. This study could potentially produce significant results that could speak to classroom practices.

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