The Transition to Advanced Mathematical Thinking: Socio-cultural and Cognitive perspectives

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This case study of the first, 'transitional' year of a mathematics programme at a research intensive university aims to deepen understandings of the transition to 'advanced mathematical thinking', or in effect, 'rigour and proof'. The case draws on ethnographic data that includes: interviews with lecturers and students; observations of tutorial/teaching sessions; a video-stimulated recall interview with a lecturer; and documents from relevant A-level and university programmes. We consider insights into transition using Activity Theory (after Leontiev, Cole, Engeström etc.) and draw on selected cognitive perspectives to Advanced Mathematical Thinking (after Fishbein, Tall, Harel, etc.). We conclude that the different activity systems of school and university involve contradictory mathematical practices and hence can result in cognitive conflicts, including those well documented by the cognitivist 'psychology of mathematics education' tradition. Drawing on these perspectives and capturing the voices of students and lecturers may be important to understanding identity, motivation, and student engagement.

Keywords: Advanced mathematical thinking, transition, identity

Introduction

This paper draws from our ongoing case study work as part of the Transmaths project, and from one case in particular. The aim of the project is to understand the transition from school/college to university study. Here we seek to explore a case of transition to 'Advanced Mathematical Thinking' from the point of view of (i) socio cultural and identity perspectives, and (ii) traditional cognitive perspectives. At the time of writing we are midway through case studies involving five universities with a focus on STEM 'programmes'.

The focus of this paper arises from one case study in a 'research university' mathematics programme, and its concern with rigour and proof especially. The School of Mathematics takes 'transition' seriously and aims to support first years in this. Yet we were told that the mathematical reasoning or 'proof' course taught throughout the first semester continues to be challenging for many of the students.

Socio-cultural and Cognitive perspectives

Harel and Sowder (2007) raise the 'socio-cultural' issues involved in the transition to Advanced Mathematical Thinking (AMT) and proof at university. However, by 'socio-cultural' they refer to curriculum and instruction, e.g. with the use of technology. This paper seeks to generalise and deepen the 'socio-cultural' to include considerations of mathematics as practice and as activity. Furthermore, we will draw on cognitive approaches to AMT, specifically the cognitive conflict students may experience during transition.

We deepen the analysis by situating mathematical cognition within Activity Theory, in the tradition of Vygotsky, Leontiev and as more recently developed by Cole and Engeström (for a full account see Williams and Wake 2007). Activity Theory views cognition as distributed in practice across systems of activity mediated by tools, social norms and social relations including the division of labour in activity. Tools include languages and concepts as well as technologies, texts and tasks. These 'carry history' and can introduce contradictions into activity (typically for instance when the tools in use are no longer fit for the purpose of the activity).

For us, then, the cognitive conflicts that students may experience in transition must be viewed as a result of a contradiction between the typical cognitions associated with pre-university and university systems and practices. In particular, we might expect to find students attempting to work in the university system using conceptual tools developed in school practices, reflecting the school system. The intuitive 'cognition' of the function concept in mathematics for instance must be understood as situated within the particular school procedural practice of graphing. By contrast, the function concept used in the university proof course is conceptual, and draws specifically in its concept definition... But we are getting ahead of ourselves.

There is substantial literature that have provided important insights into the topic of 'rigour and proof' and student learning (see Harel and Sowder 2007 for a recent summary). Essentially, these works provide an analysis of students' cognitive deficits: according to Dreyfus (1999) the reason so many students have trouble with AMT in general, and proof in particular, is that they are simply not prepared – there is a cognitive 'gap'. Others have suggested this can result in a type of cognitive conflict. Tall (1991) drew on Piaget's idea of equilibrium, and noted that when students are confronted with new knowledge which conflicts with previous knowledge, the cognitive equilibrium maybe disturbed which could result in cognitive conflict. However little insight is provided by Tall (1991) and Dreyfus (1999), and in general cognitive research to AMT, concerning how cognitive conflict may relate to other factors located in the wider socio-cultural context of the learning-teaching environment.

Furthermore, it may be argued that many students at school are prepared to treat proof – and mathematics in general – procedurally and instrumentally rather then conceptually. This may reflect a more general *'surface'* approach to learning in school: students may learn to perform instrumentally for exams rather than to think critically and approach concepts using a *'deep'* approach (Marton and Booth 1997; Williams et al. 2010).

For our research we propose that the different activity systems and practices students are expected to engage with at university which emphasises proof and the need for rigour may cause a type of cognitive conflict in students (Tall 1991). This can be framed as a conflict between old ways of being and new demands made by university practices. This can manifest itself in a misalignment between student and lecturer perspectives in learning.

Furthermore, the literature on AMT tends to emphasise learning as an individualistic process that occurs mainly on an internal cognitive level, i.e. 'in the head'. We draw on Activity Theory and relate obstacles students encounter, as well as frustrations lecturers experience as part of a wider systematic context. This includes factors related to the life-world of students such as their previous experience of mathematical practices, including approaches and motivation to learn. We also highlight other socio-cultural forces such as how certain practices and activity systems can position students/lecturers in terms of their identities in relation to mathematics. Thus, we argue that situating the literature on AMT within the framework of Activity Theory can provide insights about why students may struggle in making the transition into AMT, the effect this can have on student identity and their future trajectories in mathematics. We also consider how systems of pedagogy can be structured to help students in this transition process.

Methodology

The case study was conducted at the School of Mathematics of a research-intensive university. We employed a mixed-method qualitative ethnographic approach: this allowed for a holistic way of investigating student transition to AMT and complimented our use of Activity Theory to the research. Collecting data from a variety of perspectives allowed us to capture the different dynamics operating at various levels in learning contexts (see Jonassen and Rohrer-Murphy 1999 on methodological issues relating to Activity Theory).

Convenience sampling was used when selecting students to interview. Prospective undergraduates that intended to do a mathematics degree at our case study institution were contacted through the post and asked if they would take part in our research. Students that had agreed were contacted and interviewed over the phone or in person. These interviews involved a semi-structured format; this allowed us to question students on their first year experiences and to explore with them a number of other themes such as their reflections of how they learned and identified with mathematics prior to university. A similar format was applied when interviewing lecturers. In total six lecturers and seven students were interviewed. Each student was interviewed at three data points: (1) before or during induction week when students first joined university; (2) after students completed their first semester; and (3) when students started their second year. Observations were made during lectures, tutorials and other environments where learning took place. We video recorded six of these learning-teaching sessions, and in two cases conducted video-stimulated recall interviews with some of the lecturers taking the tutorials. Documents in the form of textbooks and worksheets were also collected. The analysis of the data collected followed a narrative and thematic approach.

Results

In the transition to university students experience a number of changes, e.g. the classrooms and learner-teacher relations provide a significant transitional ‘gap’ for many. Some lecturers are conscious of these factors, and structure support for learners early in the first year to get the students working together in groups. They also offer small tutorial sessions to support what students cover in lectures, including the course on proof. Interestingly, a mathematician (with a strong interest in mathematics education) wrote the ‘proof’ course with an awareness of much of the literature in mind. The text book illustrates a remarkable sensitivity to some of the transitional issues students encounter on a cognitive level, explicitly stating in the preface that it was designed to rectify this deficit or the ‘gap’ in their knowledge of advanced mathematics (indeed Tall’s work is cited in the text).

In making the transition to AMT we noted conflict in the activities and practices students were expected to engage with at university. From student and lecturer interviews we were informed that the type of mathematics they generally had experienced in school could be characterised as procedural, requiring little deep conceptual engagement. Thus, for many of the students interviewed, university required a different approach: one which placed emphasis on proof and rigour requiring them to engage with and critically think about concept definitions.

This different approach to mathematics resulted in a type of cognitive conflict in students, which made them question their identities as learners and influenced their future trajectories in mathematics. This cognitive conflict we found could be related to a range of factors located in the learning-teaching environment, that included a misalignment between lecturer and student meanings attributed to mathematical objects; in our case the need for proof and rigour in AMT.

Vignette 1. John's experiences with AMT

This vignette is constructed from two interviews with John: the first over the summer holiday before he started his course at university, the second after his first semester exams. The third is yet to come.

Over the summer before starting at university John told us he was eager to get stuck in. John comes from a working class family in a relatively deprived area: his school was closed and reopened; he was in the first year in the sixth form of the re-opened school and got on really well with his mathematics teacher. He converged on mathematics as his degree choice largely because he was an outstanding student. He felt he learnt to become an independent learner through the Further Maths (FM) network: he experienced lectures on visiting days at his local university, and on-line support in between. He believed he was really 'up for uni'.

By the second interview, things looked very different: the transition was a 'bit of a shock', e.g. the pace: weeks of Further Maths study were covered in one lecture (e.g. complex numbers). In addition, the 'proof' course presented a completely new view of mathematics that he really struggled with (in fact he was relieved to pass). What John experienced could described as a type of cognitive conflict: he described the proof course as very 'different', requiring a rethink about such commonplace concepts as 'counting' and 'function' that he thought he knew so well or 'clicked' in his mind.

> All I can say with the proof module, it was so new and such a different way of, just a different sort of side and view of Maths that it's a bit of shock to the system, because it didn't click straight away and it was like the first thing we did when we came to university, it was a sort of, 'hold on, we understood everything at A-level, why are we not understanding this straight away', and it was just a bit of, I guess people sort of - I probably don't like it for that reason, because it didn't click.

He was eloquent in reflecting about how A-level mathematics was reduced to 'just procedures' – even proof by induction was procedural at A-level. Looking back, he now sees some things he did wrong: he initially did not work hard enough but he now feels he has the work-social balance right. Where next? He's not certain - he has survived the shock to the system, but his view of himself as a mathematician is shaken. For John his experiences with proof lessened his confidence with mathematics: indeed he reports that he will consider taking non-proof modules in his second year. This partial story raises the question: how important are his various experiences in the trajectory of his identity? Will he veer towards or away from the kind of mathematical identity on offer, and what practices make the difference?

Vignette 2. Misalignment between students' and the lecturer's mathematical activities and practices: the case of function and its different meanings

Small group tutorial sessions are provided twice a week to support students in what they cover in lectures. These sessions involve lecturers going through the problem sheets and working with students on the solution, ideally in the form of tutor-student discussions. For this second vignette we look closely at one 'enlightened' lecturer's engagement with students' difficulties with the 'function' concept, during a tutorial. We noted that the concept definition is included A-level texts, but close examination of a typical text revealed that the school mathematical practice was essentially procedural: as we will see, the procedural experience at school set the students' dispositions in conflict with the lecturer's conceptual approach. For this lecturer, as he informed us, 'proof and the need for rigor are what define maths'.

The tutorial was spent discussing one question from the problem sheet: To find the inverse function of *f*: *R to R*, *f* (x) = 2x+3. Several of the class had answered this in a way that Robert, the lecturer, said *"Was not wrong, but likely to cause problems due to its imprecision"*. What the students had apparently done in the work they handed in was to rearrange the formula/relation y = 2x+3 to get x = ½ (y – 3), then rewritten this exchanging the x and the y to get something like y = *f* (x) = ½ (x-3).

This is a correct but arguably incomplete answer, as the domain and co-domain are left implicit (in fact both are *R* for the inverse function too, as in fact *f* is in this case a bijection from *R to R*, and it is hardly surprising that the students might therefore leave this implicit even if they thought it was important). But right or wrong, why is this method such a powerful attractor for the students? In practice, in school activity systems – a function is always described as a 'rule' mapping from x-values to y-values, represented as real number lines horizontally and vertically, most often as a graph. In this sense it can be described as a powerful concept image, or better prototype (Fishbein 1990). However, a social practice perspective would assert merely that this is how the students are 'schooled': this is typically what they practice for school exams.

Robert seems to have selected this example to discuss because he had marked their work beforehand and saw an opportunity to emphasise the need for complete, explicit writing out of the mathematical proof here (with explicit definition of the sets X and Y). He therefore came to the supervision knowing the students work and what he wanted to achieve in *'deconstructing and reconstructing'* their informal mathematics using his preferred degree of rigour and precision. Robert proceeded to write down how he would prove *f* is a bijection, starting with the definition of an inverse in which the domain and co-domain are explicitly included, thus:

> Definition: the inverse function of *f*: X to Y [that is, a function that maps any x in X to a single value y in Y given by a rule y= *f* (x)] is a function g that maps a value y in Y to a value x in X, (i.e. g(y) = x) such that g *f* (x) = x for all x in X.

He began to work through the algebra but was interrupted by a student who said he had the answer right his own way (as mentioned above). Robert stated *"It is not wrong but why exchange the values x and y?"* which he claimed was likely to lead to confusion of the domain and co-domain. Robert encouraged the same student to illustrate the reasoning behind his answer on the board. Indeed, most of the class seemed to agree *"That is the way we were taught at A-level"*.

The student then fortuitously chose to exemplify his approach using exponential functions (from a later worksheet question): he argued that the expression y = exp(x) was the same as x = log(y), and that switching x and y was required to 'get' the inverse function y = log(x). Implicitly we assume the graph of y = exp(x) is the same as that of x = log(y), and so does not represent the inverse. However, Robert noted *"Oh yes it might be, if you consider the graph as a representation of the function mapping y to x"*. The idea that the same graph 'represented' two inverse functions however seemed counter-intuitive – but drawing on Activity Theory, with a focus on practice, we say rather it is counter to all school practices of mathematics the students had experienced hitherto.

Why is this serendipitous example pedagogically 'useful'? Because unlike the worksheet example the domain and co-domain of the inverse function log(x) is problematic: since the exponential function has no negative values, log(x) has no values for negative x (within the Reals anyway). Robert was therefore able to point out how important the specification of the domain and co-domain are to this work. An intuitive approach would have simply led one to say log and exp are inverse functions without specifying the functions rigorously (i.e. completely and precisely).

Analysis of two mathematical activity systems and practices at work

This case study highlighted a conflict between school activity systems and practices, and the 'rigour/precision' required in the university mathematics activity system and practice: it rests in one sense in the contrast between informal procedures in school and rigorous conceptual work of pure mathematics at university.

From the second vignette we can note that the students believe exp(x) and log(x) ARE inverse functions, whereas Robert pointed out that these expressions are NOT functions, but values. Students would have been introduced to the idea that the graphs of f and its inverse g would be reflections/images of each other in the line y=x. The suggestion that the one graph can be thought to represent two functions (y as a function of x and its inverse x as a function of y) would be counter to school practice – and what cognitivism might say is counter-intuitive. In Activity Theory terms, these 'cognitive intuitions' are embedded in unexamined 'operational conditions' that fall below the level of consciousness, and are all the more powerful for being unexamined and uncritically accepted. We can view school mathematics as a practice developed within the AS/A2 of the school/classroom/curriculum; what we are seeing is a conflict between the old ways of being and the new demands caused by Robert's subjective view of mathematical objects (drawing on university/academic practices). Objects, like 'functions', have contested meanings here because they are boundary objects used in two quite different practices and systems – that of the school and the university. This contradiction might suggest that the school activity system, or maybe the university activity system, should be modified or re-designed to help better align these practices: to shed light on the conflicts for the benefit of both learner and teacher. In a sense the dialogue between Robert and his students can be thought of as just such a re-design.

Analysis of the pedagogy/curriculum

The fact that the tutorial was based on the students' work (previously marked) and that Robert brought this prominently into the session may have triggered what happened: making the intuitive and implicit approaches taken by the students more explicit. We have rarely seen this Assessment-for-Learning style pedagogy in our other university case studies (usually the lecturer uses the postgraduate assistants to mark the work, and thus misses a great opportunity to learn about their students' mathematical practices/errors). This is difficult to envisage in a lecture with large student numbers, and the ensuing pre-conceived 'coverage' and norms of interaction.

Even though Robert was aware of the vagueness and unhelpfulness of the informal school approach to learning AMT, he was unaware of the depth of the clash of values and practices involved in this one example. He might have said that he wanted them to *"write their solutions out properly"* with attention to precise definitions, etc, but had not explicitly realised that his students' practices with regard to function did not take account of its concept definition. Thus the most important element of learning derived from the interaction between Robert and his students might be his enhanced understanding of the implicit beliefs they brought to their learning. By confronting their mathematical practices he was able to impart both a deepened understanding of AMT and cause them to reflect on the cognitive conflicts which may occur when old knowledge conflicts with new ways of understanding mathematics (Tall 1991). This kind of learning is one that Robert endorses: he is a keen and enthusiastic teacher as well as a research academic. After our video-stimulated recall interview he says that he has learnt some things about the way students think and have been influenced by schooling, and sees its potential use for him as a lecturer.

Concluding Remarks

The cognitive perspective helps us understand the transition to Advanced Mathematical Thinking as a form of correction of a cognitive deficit. Furthermore, we argue that viewing the 'conceptual' transition to AMT as part of a shift from the practices and activities of one system to another raises a broader range of issues: these practices are shaped by more than only the 'thinking' that goes on in the head, but also by the learning approach, a set of priorities, expectations and dispositions, a set of tools, social relations and norms of behaviour too. These are all important aspects to consider. Thus, it could be argued that the predominant practice with graphs and functions in school mathematical practices (mainly focusing on graphing) leaves the specification of domain and co-domain irrelevant, even though A-level text books define them as essential to the concept. More broadly, proving is perceived by students as practising a set of instrumental procedures: it has never been practised as a social process required to persuade a critical audience beyond reasonable doubt. We have particularly seen how the teaching and learning practices may be an essential part of the transition problem/solution.

Finally we have alluded to the issue of identity. For the learner (especially noted with John above) transition may be a threat to an existing or emerging mathematical identity, or conversely it may be an opportunity to re-invent himself as a different kind of mathematician. The resolution of this will, in our view, be critical for the way John engages in his future studies (we currently await the third interview with interest). However, identity is a crucial issue for Robert also: 'rigour and proving' is for him what constitutes the essence (the pure episteme) of mathematics. Yet he also identified himself as a teacher: he wants to take the students with him across the gap (the threshold), while at the same time acknowledges that for many students this will not happen.

We can see the contradictions between school and AMT practices, or between schooling and university activity systems being lived out in the mutual engagement of these identities in the tutorial: it is manifested as a tension between the subjective 'ways of being' a mathematics student and a lecturer. These tensions can be constructive as they bring to the fore different discourses and what it means to be a mathematician and a learner of mathematics. This could be incorporated into the design and pedagogy of the curriculum: but there are so many obstacles to this, not least the macro- and meso-economic and political structures in the systems of schools and universities, but also in the historically constituted professional identities of 'homo academicus'.

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