Developing early algebraic reasoning through exploration of the commutative principle

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Student transition from arithmetical understandings to algebraic reasoning is recognised as an important but complex process. An essential element of the transition is the opportunity for students to make conjectures, justify, and generalise mathematical ideas concerning number properties. Drawing on findings from a classroom-based study, this paper outlines how the commutative principle provided an appropriate context for young students to learn to make conjectures and generalisations. Tasks, concrete material and specific pedagogical actions were important factors in students' development of algebraic reasoning.

Introduction

For those students who complete their schooling with inadequate algebraic understandings access to further education and employment opportunities is limited. An ongoing concern for these students in New Zealand and internationally, has resulted in increased research and curricula attention of the teaching and learning of algebraic reasoning. To address the problem one response has been to integrate teaching and learning of arithmetic and algebra as a unified curriculum strand in policy documents (e.g., Ministry of Education 2007, National Council of Teachers of Mathematics 2000). Within the unification of arithmetic and algebra, students' intuitive knowledge of patterns and numerical reasoning are used to provide a foundation for transition to early algebraic thinking (Carpenter, Franke, & Levi 2003). Importantly, this approach requires the provision of opportunities for students to make conjectures, justify, and generalise their mathematical reasoning about the properties of numbers.

Carpenter and his colleagues explain that deep conceptual algebraic reasoning is reached when students engage in "generating [mathematical] ideas, deciding how to express them …justifying that they are true" (2003, 6). We know, however, from exploratory studies (e.g., Anthony & Walshaw 2002, Warren 2001) that currently many primary age students have limited classroom experiences in exploring the properties of numbers. These studies illustrated that more typically students experience arithmetic as a procedural process. This works as a cognitive obstacle for students when later they need to abstract the properties of numbers and operations. These studies also investigated student application of the commutative principle and illustrated that many students lack understanding of the operational laws. Both Anthony and Walshaw's study of Year 4 and Year 8 students and Warren's studies involving Year 3, Year 7 and Year 8 students demonstrated that many students failed to reach correct generalisations regarding commutativity. The students recognised the commutative nature of addition and multiplication; but also thought that subtraction and division were commutative. Anthony and Walshaw showed that although students offered some explanation of the commutative property none offered generalised statements nor were many students able to use materials to model conjectures related to arithmetic properties. These researchers concluded that very few students were able to draw upon learning experiences which bridged number and algebra.

Nevertheless studies (e.g., Blanton & Kaput 2003, Carpenter et al. 2003) which involved teaching experiments provided clear evidence that young children are capable of reasoning in generalised terms. These studies illustrated that they can learn to construct and justify generalisations about the fundamental structure and properties of numbers and arithmetic. Importantly, they demonstrated that when instruction is targeted to build on students' numerical reasoning they can successfully construct and test mathematical conjectures using appropriate generalisations and justifications.

Theoretical framework

The theoretical framework of this study draws on the emergent perspective promoted by Cobb (1995). From this socio-constructivist learning perspective, Piagetian and Vygotskian notions of cognitive development connect the person, cultural, and social factors. Therefore, the learning of mathematics is considered as both an individual constructive process and also a social process involving the social negotiation of meaning.

I draw also on the body of research that suggests that making and representing conjectures, generalising, and justifying are fundamental to the development of algebraic reasoning (Kaput 1999). For young children the development of early algebraic thinking needs to go beyond simply making conjectures. Children need to gain experience in using mathematical reasoning to make explicit justifications and generalisations. Constructing notations for representing generalisations is also an important part of the generalising process (Carpenter et al. 2005). Carpenter and Levi (2000) promote using number sentences to provide students with access to a notational system for expressing generalisations precisely. Also, number sentences provide a context whereby students' implicit knowledge becomes explicit.

Whilst students' propensity to offer justifications can be encouraged by classroom norms that reinforce the expectation that justifications are required, providing adequate mathematical explanations requires appropriate scaffolding, modelling and teacher intervention (Carpenter et al. 2005). Studies (e.g., Carpenter et al., Lannin 2005) which have examined the forms of arguments that elementary students use to justify generalisations classify students' justification as either empirical or generic examples. In the first instance, most students view specific examples, or trying a number of cases, as valid justification. These and other studies (e.g., Kaput 1999) have shown that using concrete material can support young students to develop their justification skills. Therefore the purpose of this paper is to report on how an examination of the commutative principle offered young students a valuable context in which to learn how to make conjectures and construct generalisations. A particular focus is placed on the role of mathematical inquiry, concrete materials, and teacher interventions which scaffold the students to use arithmetic understandings as a basis for early algebraic reasoning. The specific question addressed in the paper asks: How can the exploration of the commutative property of numbers support students to use arithmetic understandings as a basis for early algebraic reasoning?

Methodology

This research reports on episodes drawn from a larger study which involved a 3-month classroom teaching experiment (Cobb 2000). The larger study focused on building on numerical understandings to develop algebraic reasoning with young students. It was conducted at a New Zealand urban primary school and involved 25 students aged 9-11 years. The students were from predominantly middle socio-economic home environments and represented a range of ethnic backgrounds. The teacher was an experienced teacher who was interested in strengthening her ability to develop early algebraic reasoning within her classroom. Each lesson followed a similar approach. They began with a short whole class introduction, then the students worked in pairs and the lesson concluded with a lengthy whole class discussion.

At the beginning of the study student data on their existing numerical understandings was used to develop a hypothetical learning trajectory. Instructional tasks were collaboratively designed and closely monitored on the trajectory. The trajectory was designed to develop and extend the students' numerical knowledge as a foundation for them developing early algebraic understandings. This paper reports on the tasks on a section of the trajectory which built on student understanding of commutativity as a context which supported their algebraic reasoning. The students were individually pre and post interviewed using a range of tasks drawn from the work of other researchers (e.g., Anthony & Walshaw 2002, Warren 2001). The rationale for selecting these questions was to replicate and build on the previous findings of these researchers. Other forms of data collected included classroom artefacts, detailed field notes, and video recorded observations.

The findings of the classroom case study were developed through on-going and retrospective collaborative teacher-researcher data analysis. In the first instance, data analysis was used to examine the students' responses to the mathematical activity, and shape and modify the instructional sequence within the learning trajectory. At completion of the classroom observations the video records were wholly transcribed and through iterative viewing using a grounded approach, patterns, and themes were identified. The developing algebraic reasoning of individuals and small groups of students was analysed in direct relationship to their responses to the classroom mathematical activity. These included the use of concrete materials, the classroom climate of inquiry, and the pedagogical actions of the teacher.

Results and discussion

I begin by providing evidence of the initial understandings of the students. I then explain the starting point for the section of the trajectory related to the commutative principle. The initial starting point for classroom activity is outlined and I explain how this was used to press student reasoning towards richer understandings using concrete representations. Explanations are then offered of how the press toward deeper student reasoning was maintained through the introduction of symbolic notation. I conclude with evidence of the effect of the classroom activities using post student interview data.

Interview data of the initial concepts of the commutativity principle of the students

This section presents the initial task interview results. True and false number sentences (see Figure 1) were used to explore student understanding of the commutative principle.

15 + 6 = 6 + 15	15 – 6 = 6 – 15
15 x 6 = 6 x 15	15 ÷ 6 = 6 ÷ 15

Figure 1: True and false number sentences

Twenty of the twenty five students participating in the study could not identify which number sentences were true or false. Many considered that they were all true which confirmed that they had limited understanding of the commutative property of addition and multiplication. Five of the twenty five students could identify which sentences were true. However none of these students were able to provide further explanation or justification for their reasoning.

The stepping off point on the trajectory

In order to focus student attention on the correct application of the commutative principle, in an initial activity the students worked in pairs to identify true and false number sentences. The data illustrates that the students readily recognised that addition number sentences were true (e.g., 15 + 3 = 3 + 15 and 5 + 6 = 6 + 5).

> Hamish: It's just the same equation spelt backwards.
>
> Matthew: Three plus fifteen is just fifteen plus three twisted around so it is exactly the same.

However the commutative principle of multiplication posed more challenges. As an example, in an initial lesson, one group of students concluded that the commutative law only applied to addition. During a discussion a student stated:

> Ruby: Six times five equals more than five times [six] so it wouldn't work in that way.

Another student supported her argument with an erroneous example:

> Hamish: One times zero is zero and zero times one is one.

Although the students had begun to justify their reasoning using additional examples it was evident to the teacher and I that they needed to extend and deepen their reasoning. This was particularly so if they were to learn generalise the numerical relationship between addition and multiplication.

Using representational material to press the reasoning

Collaborative discussion led to revision to the trajectory and the insertion of other mathematical activities. It was evident to us that the students did not have access to representations on which they could base their mathematical explanations of the commutative law. Therefore, we placed an explicit focus on the use of a range of different equipment. This offered the students ways to justify their conjectures, and shift their arguments into more generalised terms.

The students working in pairs, using the true and false number sentences as a basis for discussion, were asked to formulate conjectures about the commutative properties of addition and multiplication. Equipment (popsicle sticks, and counters) was introduced. The students were required develop explanations but also to represent and justify their conjectures using the concrete materials. The teacher carefully scaffolded the explanations so that the students integrated their verbal statements and justifications using concrete materials. But she also ensured that the students were pressed beyond the use of concrete materials, or further verbal examples to more generalised reasoning. For example in the following whole class discussion the teacher selected students to model how they represented and explained their conjectures for 3 + 15 = 15 + 3 using popsicle sticks.

> Hannah: [swapped the pile of three sticks with the pile of fifteen sticks] Three plus fifteen equals eighteen but you could just swap the other ones like the fifteen with the three and the three with the fifteen so it does equal eighteen.

The teacher then revoiced Hannah's statement to develop a more specific explanation of the commutative nature of addition:

> Teacher: So you are saying that if you just swap them around it will still be exactly the same amount?

She then shifted the discussion into general terms and facilitated all the students to consider a more generalised understanding:

> Teacher: Would that work for any set of numbers then when you are adding?

On the trajectory we had considered the need to consider deepening the students' understanding of multiplication specifically. Other forms of equipment were introduced to support the students to visualise the commutative property of multiplication. These included the use of animal arrays (pictures of animals) and counters used in an array. As a result the students became facile in the use of counters to justify the commutative nature of multiplication as evident in the explanation:

> Sabrina: [builds an array with counters] We put four down there and then we did five across…we thought that if you turn it around and put them down here, it is the same five rows of four.

Evidence is provided in the data that mid way in the study the students were now able to provide appropriate mathematical explanations for the commutative property for addition and multiplication and justify their reasoning using concrete representations.

At the same time we were aware from the data collected in the initial interviews that the students over-generalised commutativity to include subtraction and division. At this mid-point in the study specific true and false number sentences were designed and used which extended beyond addition and multiplication to include subtraction and division. These were used with materials to prompt student exploration of the commutative principle with other operations. The teacher closely observed student activity and probed student reasoning:

> Teacher: Does it work with other things like division or subtraction?

Through lengthy discussion which caused conflict for many students they began to provide clear explanations of the non-commutative nature of subtraction and division. Justifications were provided most often as a counter-example as illustrated in the following student's response:

> Gareth: Seven minus four doesn't equal four minus seven… because seven minus four equals three and four minus seven equals minus three.

Constructing clear understandings of the commutative properties of addition and multiplication formed a foundation for their explanations of why the commutative property did not operate for subtraction and division.

Shifting the press to representing conjectures symbolically

In line with the progression on the trajectory we now analysed that the students were ready to use symbolic notation to further press towards generalised reasoning. The teacher scaffolded the use of symbolic notation during a whole class discussion after the students had used both materials and notation to represent their conjectures. She asked them to refer to the conjectures they had recorded while using the material to explore the commutative property and to use an algebraic number sentence to represent these:

> Teacher: Can you write this as a number sentence that would be true for any number?

Analysis of the data illustrates that in response, many students readily provided a range of algebraic number sentences. For example, the following students said:

> Susan: Z times Y equals Y times Z.
>
> Steve: A B equals B A
>
> Gareth: We did rectangle plus B equals B plus rectangle.

In accord with the trajectory, in the following lesson the teacher further promoted generalisation of the commutative principle through a discussion of symbolically represented conjectures. She recorded symbolised conjectures (see Figure 3) on the white-board.

B + ■ = ■ + B
J + T = T + L
Q x R = R x Q

Figure 3: Symbolised conjectures

Students were then asked to discuss the symbolised conjectures:

> Teacher: Can you look for the ones which are always true…think about why it is always true as well?

In response to the teacher prompt the students illustrated their knowledge that addition was always commutative:

> Heath: They always will be true... because they are just a reflection…
>
> Ruby: It's just swapped… it is just the same numbers the opposite way.

Another student used the symbolised conjecture to make a general statement about the commutative law:

> Sangeeta: If you use two sets of numbers which are the same the statement will always be true… with addition or multiplication.

In this way, number sentences both provided notation for the students to represent their conjectures and facilitated them to develop more proficient generalisations about the commutative property.

The combination of extended exploration, the requirement that students explain and justify their reasoning with materials and the press for more generalised reasoning using symbolic notation, supported the students to provide more proficient explanations of the commutative principle. This is evident in the following statement:

> Josie: If you get two numbers and you times them by each other…and then if you times them by each other the other way around it will always be the same answer.

Interview data of understandings of the commutativity principle post-study

At the concluding interview nineteen of the twenty-five students correctly applied the commutative principle to addition and multiplication. Appropriate justification was provided using concrete models. For example, one student using an array explained:

> Josie: Because if you swap it around it is like 12 groups of 4 or 4 groups of 12 which is the same.

She then extended the explanation and drew an array of six times five (see Figure 2):

> Josie: Because you can put it into groups that way or that way and it always works.

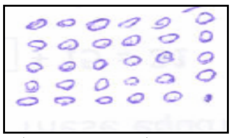

Figure 2: Josie's array

Other students constructed number sentences which generalised the relationship using symbolic notation.

> Steve: J plus C equals C plus J … because you can always reverse stuff in adding.

Six of the twenty five students over-generalised the commutative principle to include subtraction and division. These students had participated in the whole class activities and evidence is provided in the data that during collaborative group work activities they were able to recognise the non-commutative nature of subtraction and division. However without the scaffolding of group activity, in the interview process they reverted to over-generalising the commutative properties.

In the final interview, the way in which most students represented the commutative principle using mathematical explanations, representations, and justification confirmed that the tasks, pedagogical actions, and the classroom environment had scaffolded student understanding of the commutative property.

Conclusion and implications

This study sought to explore how student exploration of the commutative principle deepened their understanding of arithmetic properties whilst also supporting their construction of conjectures, justification and generalisations. Similar to the findings of Anthony & Walshaw (2002) and Warren (2001), many of these students initially failed to reach correct generalisations regarding commutativity. Extending the task beyond true and false number sentences and the introduction of equipment led to student modelling of conjectures and provision of concrete forms of explanatory justification. Importantly, teacher interventions were required to shift students to make generalised statements about the commutative principle.

Results of this study support Carpenter and Levi's (2000) contention that use of number sentences provides students with access to a notational system for expressing generalisations precisely. The symbolic representation of their conjectures coupled with the use of equipment and teacher press for generalisation led to more specific student generated generalisations.

Many of the students in this study deepened their understanding of arithmetic properties. However, the small proportion of students who continued to over-generalise to include subtraction and division indicate the need for multiple opportunities over an extended period of time for students' to develop deep understanding of operational laws.

Findings of this study affirm that the context of the commutative principle can provide students with effective opportunities to make and represent conjectures, justify and generalise. Appropriate tasks, concrete material and teacher intervention supported students to develop their understanding of the commutative principle. Opportunities to develop explanations with concrete material and use notation to represent conjectures led to students developing further generalisations. Due to the small size of this sample further research is required to validate the findings of this study.

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