

## **Reasoning on transition from manipulative strategies to general procedures in solving counting problems**

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We describe the procedures used by 11- to 12-year-old students for solving basic counting problems in order to analyse the transition from manipulative strategies involving direct counting to the use of the multiplication principle as a general procedure in combinatorial problems. In this transition, the students sometimes spontaneously use tree diagrams and sometimes use numerical thinking strategies. We relate the findings of our research to recent research on the representational formats on the learning of combinatorics, and reflect on the didactic implications of these investigations.

### **Introduction**

The work on combinatorics at school is restricted in many cases to the use of formulas that limit the development of reasoning. English (1991) and Fischbein and Gazit (1988) emphasised the interest of students' reasoning processes when solving combinatorial problems and their educational implications. At the time of such investigations, Piagetian theory affirmed its relevance to cognitive psychology, considering combinatorics as an essential component of reasoning. Currently, the interest in discrete mathematics and, particularly in combinatorics is increasing the research on this content (Jones, 2005).

Much previous research related to our interests has focused on early education, and the detected strategies emerge from a context where students solve counting problems whose solution is usually a number small enough to be obtained by enumerating all possibilities, and counting one by one afterwards. (Empson and Turner, 2006; English, 1991; English, 1993; Steel and Funnell, 2001). Within this context, English (2007) concludes that 7- to 12-year old students "with no prior instruction and receiving feedback only through their interaction with the physical materials, the children were able to apply their informal knowledge of the problem domain to their initial solution attempts." (p. 152) She suggests that activities with tree diagrams and systematic lists lead 11- and 12- years old children to derive the basic formula for combinations (p. 154).

In this paper, we analyse the case of five selected students of this age to explore how they use their previous knowledge to develop strategies leading to the use of the general, and highlight how numerical reasoning of the students arises using sophisticated representations different than tree diagrams. Consequently, we call into question attempts to start directed instruction of tree diagrams at a too early stage.

### **Reasoning strategies and scheme in counting problems**

English's investigations (English, 1991, 1993, 2007) allowed characterising the strategies used by children between 4 and 12 for counting the possible arrangements of two and three elements. The identified strategies grant an important role to

manipulation and may be arranged in increasing order of complexity, from a resolution based on a trial and error approach to the odometer strategy.

For more sophisticated tasks, Fischbein and Grossman (1997) refer to a scheme as a program that allows the problem solver to interpret a certain amount of information and prepare the corresponding reaction. They consider that the procedure performed by students leads them to compute the solution to a counting problem as a scheme that allows expressing the total number of possibilities. The formulae for calculating the number of  $n$ -permutations or  $(n,k)$ -combinations are examples of schemes. The work carried out on combinatorial reasoning by Fischbein and Grossman (op. cit.) and Kolloffel et al. (in press) recognise four basic patterns (schemes) of counting for solving these kinds of problems:

- Permutations of  $n$  elements.

- Arrangements without replacement: Number of selections of  $k$  elements which one may obtain from  $n$  given elements, considering that no element may be used more than once in a selection and that the order of elements is relevant.

- Arrangements with replacement: Number of selections of  $k$  elements which one may obtain from  $n$  given elements, considering that every element may be used more than once in a selection and that the order of elements is relevant.

- $k$ -Combinations of  $n$  objects

These schemes have immediate didactic consequences, as combinatorial problems used to be classified also under four types, emphasising the presence of one of those schemes, which, once discovered, allow the student to find the solution. However, many combinatorial problems do not admit an approximation by these schemes because the criteria of repetition and order are not obvious. Moreover, many students fail to solve problems where those schemes must be modified or when the statement of the problem requires to reproduce the steps involved in the scheme construction. This fact led us to go back to the much more basic scheme involved in combinatorial formulae, namely the multiplication principle, which can be stated generally as follows:

The total number of arrangements of  $k$  elements having  $n_1$  possibilities for the first one,  $n_2$ , for the second and in general  $n_k$  for the  $k$ -th, each position being independent of the others, is  $n_1 \cdot n_2 \dots n_k$ .

The multiplication principle of two sets is the simplest scheme that underlies those of permutations and arrangements with and without replacement, and together with basic arithmetic operations leads to that of combinations. Thus, we focus our attention on the way how students acquire this basic scheme by themselves, and pay special attention to the use of different representations. We use the theoretical concept of representational format as used by Kolloffel (2008).

### **Representational formats in the combinatorial problem solving**

From a cognitive perspective, Holyoak and Morrison (2005) emphasise the relationship between problem solving and representations performance of subjects and conclude that the representation used to solve a particular case is a key factor in solving the general problem. Rico (2009) characterised the notion of representation as all those tools -signs or graphics- which are present mathematical concepts and procedures and with which the subjects dealt with and interact with the mathematical knowledge, i.e., record and communicate their knowledge about mathematics.

There is agreement in mathematics education to distinguish between internal and external representations. Although both types of representation should be seen as separate domains, from the genetic viewpoint, external representations are

characterised by acting as a stimulus for the senses in the process of building new mental structures and allow the expression of concepts and ideas to individuals who use them. Ideas must be represented externally in order to communicate them (Hiebert and Carpenter 1992). We focus our attention on the external representations as those that have a trace or tangible support even when this support has a high level of abstraction (Castro and Castro 1997).

In the specific case of combinatorics, Kolloffel et al (2008) focus their research on three representational formats: (a) arithmetic, (b) text and (c) diagrams. Diagrams are considered to help students to understand new situations. Its functionality and, particularly, tree diagrams, has been analysed in several studies. For example, Fischbein and Gazit (1988) give the maximum benefit to the tree diagrams in its instructional programs. They consider that tree diagrams are representative of a state of maturity in the counting strategies. The effectiveness of this type of graphical representation has been questioned by Kolloffel (op.cit), arguing that the benefit is restricted only to conceptual learning, and taking into account the possibility of combining two or more representations. Moreover, in our investigation, we will deal with a new category of representational format which also integrates two or more representations, but under the additional condition that none of them by themselves make sense of the problem. We'll call this new type of representation synthetic representation.

### **Research objectives and methodology**

Our research aim is to describe how students become able to use the multiplication principle in the context of a combinatorial problem. This general objective is broken down into two specific objectives:

- To characterise the strategies used for solving combinatorial problems which solution is a number larger enough so that the students can not calculate it by enumerating all possibilities and then counting them one by one.
- To characterise the process of solving a problem leading to generalize the multiplication principle in terms of the representation used.

We selected five cases from a previously selected sample of 25 students. Previous research on children's strategies for solving combinatorial problems (English, 2007) informed the first decision on the selection of this sample from a cognitive point of view, that was consider a group of students about 12 years. On the other hand and in order to minimize other contextual variables, it was decided to gather data under optimal conditions for the students' involvement and interest for the activity. This informed a second decision on the selection of the sample. In respect of it and to make it clearer to the reader, we need to expose briefly about a national project seeking to stimulate mathematically talented students. This project's main objective is to identify, guide and stimulate interest of students aged 11 and 12, who are particularly attracted by the beauty, depth and usefulness of mathematics (Hernandez and Sanchez 2008). Mathematic teachers are informed through teacher's associations about the project and asked to propose possible candidates to joint it. There are also public calls in newspapers and the internet. From about 300 candidates, only 25 are selected by a test of mathematical problem solving and by interviews to ascertain their interest in participating. Selected students show a good aptitude and attitude toward mathematics, but they are not necessarily gifted.

The group of 25 students participating in the project last year became our research sample and data were collected during the programmed sessions within the context of the project. They had not worked previously in combinatorics and were at

the beginning of the first three sessions devoted to this subject. The objectives for these sessions included reasoning and deducing counting methods, and the first step was to inquire about the multiplication principle. In this paper, we focus on analysing the first three questions proposed (Figure 1), as they were especially oriented towards obtaining information about student's depth of knowledge on the multiplication principle. We asked them to elicit all their actions by writing and then analysed students' reports on the solution of the problem. For the selection of cases to be analysed, the first step was to consider if question 1 was solved correctly or not, and if the correct solutions made use of the multiplication principle immediately. Cases would be interesting for us if the students solved the problem correctly but did not immediately use that rule.

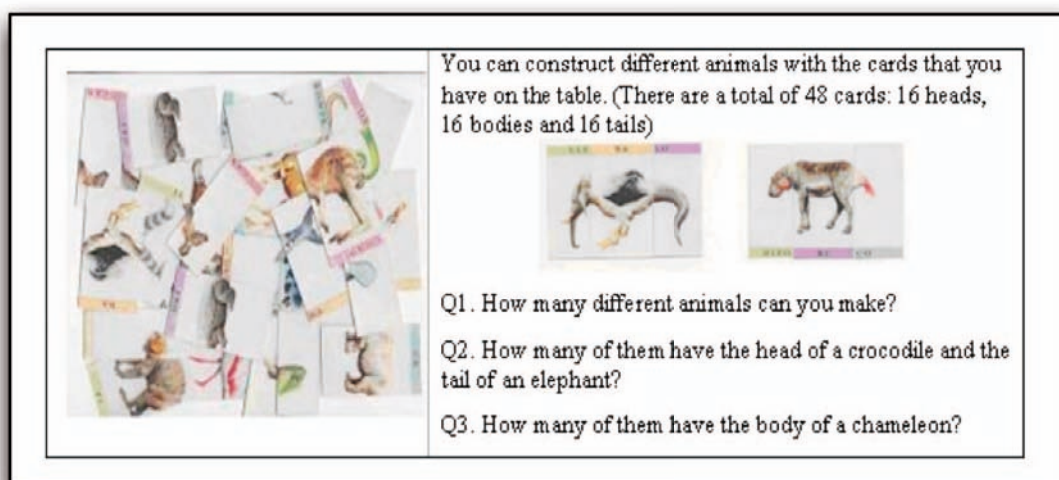


Figure 1: Proposed problem with three questions. The cards were provided to the students.

Twenty students solved the problem by directly multiplying  $16 \cdot 16 \cdot 16$  and elicited no more actions, but five did something different. These five cases were considered for further research. The fact that in a more or less homogeneous group in terms of age, interest and capacities most of the students use immediately the multiplication principle, allows us to assume that those who do not do it are still in the process of consolidate it. This justifies why the selected cases were considered especially relevant to our investigation.

### **A bird's eye on the selected cases**

#### ***Ana***

Ana uses a tree diagram to represent the statement, which permits her to organise all possibilities (see Figure 2). In order to create this diagram, Ana sets a first card for the animal's head and covers all possibilities for this. For the particular case of two cards in the second position, the student represents all possible options. Following the categories proposed by English (1993), Ana is using the most effective exhaustive strategy for counting.

The tree diagram allows the student to represent the easier two dimensional problem ( $1 \cdot 16 \cdot 16$ ). Then Ana introduces a textual representation of this particular case, which permits her to identify a multiplicative pattern expressed arithmetically ( $1 \cdot 16$ ),

thus solves the two-dimensional problem, and leads to extend the tree diagram to analyse the dimensional situation -16-16. Again, a textual representation leads to represent the solution mathematically:  $16$  ( $2^{\text{nd}}$  position)  $\times$   $16$  ( $3^{\text{rd}}$  position)  $\times$   $16$  ( $1^{\text{st}}$  position), following the order reflected in the tree diagram.

Ana also comes to use a symbolic representation of three-dimensional patterns. After solving the problem, the solution of questions 2 and 3 is given immediately by multiplying the number of possibilities for each position and we could infer that the multiplication principle has been interiorised.

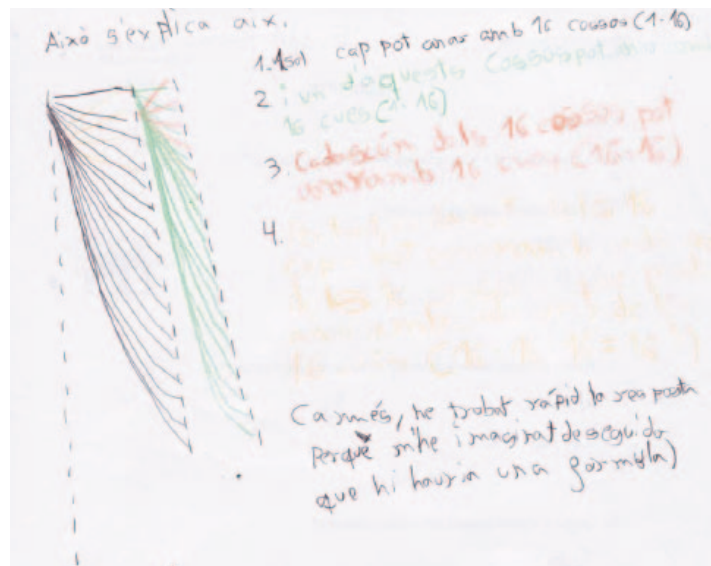


Figure 2. Tree Diagram used by Ana. (Translation: 1. Each head can combine with 16 bodies // 2. And one of these bodies can be combined with 16 tails // Each one of the 16 bodies can be combined with 16 tails // Then each of the 16 heads can be combined with each one of the 16 bodies, which can be combined with the 16 tails.

### ***Biel***

Biel starts by solving the particular problem in which the number of cards for each item is 3. In contrast to the previous student, Biel has taken this step without a prior external representation. Then, he uses a comprehensive representation of all possibilities through the tree diagram that allows counting all options one by one, and leads him to detect some pattern, which is expressed as  $3^3$ .

After solving an easier problem using cards, Biel generalises his arithmetic representation of 3 cards to the problem with 16 cards, and writes  $16^3$  as the solution of the proposed problem. Then he also expresses immediately the solutions to questions 2 and 3 in terms of multiplication of possibilities for each position.

### ***Carles***

Carles identifies a multiplicative pattern in the statement and write down two conjectures:  $16 \cdot 3$  and  $16^3$ . Afterwards, the student proceeds to justify them. To this end, Carles uses a representation of the conjecture  $16^3$  in which we find elements of a tree diagram and a textual representation. Each form of representation does not give meaning to the problem independently, but taken together, they do, what reinforce our theoretical position about consider a new category of representational format integrating two representations.

After obtaining an answer to the problem, Carles tries to refute the guess  $16 \cdot 3$  with a textual representation of the operation in the context of the problem:

The 3 are the parts they have, but have no relationship,

thus suggesting that the multiplicative pattern that responds to a sum repeated 3 times does not fit the problem. As Ana and Biel did, after the solution of the first question, Carles immediately expresses the solution of questions 2 and 3 using the multiplication principle.

### *David*

David addresses the problem by combining six cards belonging to two real animals. He solves it by counting one by one 8 animals. He then seeks to solve the problem for the case of three real animals by extending the previous result, and writes down the following false conjecture:

With the three selected animals, 3 different pairs of cards can be made. For each pair 8 possibilities were obtained [before], and therefore the total number of possibilities will be  $8 \times 3 = 24$ .

Furthermore, he counts one by one the 27 possibilities for this case, what leads him to refute the initial guess. Thereafter, David uses a synthetic representation (textual-arithmetic) that allows him to argue what was wrong.

Finally, he tries again to look for a new pattern from the particular cases of two and three animals, and finds the correct answers  $2^3$  and  $3^3$ . From that concludes the correct answer  $16^3$ . However, in contrast to Ana, Biel and Carles, he fails to identify the multiplication principle to solve the second and third question immediately.

### *Eva*

Eva's reasoning is similar to that of David, beginning with one particular problem, guessing three different conjectures, and looking for their justification or refutation through arithmetical representations (Figure 3). She also faces the solution of questions 2 and 3 as new independent problems.

## **Discussion of results**

A first approach to the work of these five students permits us to identify that three of them (Biel, David and Eva) start by a certain inductive process of reasoning, while the other two (Ana and Carles) use a specific representation of the problem. Based on this initial difference, we have identified the different steps that each of the students performed, and the representational formats they used in solving problems (see Table 1). David and Eva show a very rich inductive process from a heuristic point of view. This is very effective for reasoning in mathematics, and allows them to effectively solve the first question. However, they do not respond directly to questions 2 and 3 in terms of the multiplication principle as the others do. Their reasoning enables them to generalise about the number of possibilities for each of the three placements (16) but not about the number of placements. As a consequence, they do not generalise the multiplication principle. David and Eva do not use tree diagrams.

In contrast, the reasoning process followed by Ana, Biel and Carles does lead to generalise the number of placements. Focusing on the students who use the tree diagram, its use is more effective for Biel, who uses this diagram to represent a particular case. In this sense, we suspect that the failure to address the problem by working with individual cases increases the effectiveness of this representation for the generalisation of the multiplication principle.

Table 1. Each cell shows the representational format used at different steps of students' reasoning.

	Cases				
	Ana	Biel	Carles	David	Eva
Complete representation of the statement	Graphic (diagram)		Arithmetic		
Study of particular cases	Synthetic (Arithmetic - textual)	Graphic (diagram) Arithmetic		Synthetic (Textual-Arithmetic)	Arithmetic
Organisation of particular cases				Synthetic (Textual-arithmetic)	Arithmetic
Pattern detection	Arithmetic	Arithmetic	Arithmetic	Synthetic (Textual-arithmetic)	Arithmetic
Conjecture			Arithmetic	Synthetic (Textual-arithmetic)	Arithmetic
Justification of the conjecture			Synthetic (Graphic-textual)	Synthetic (Textual-arithmetic)	Arithmetic
Generalisation	Algebraic	Arithmetic	Arithmetic		

## Conclusions

This research responds to theoretical and teaching interests. From a theoretical point of view, it provides specific information concerning the use of inductive reasoning in solving combinatorial problems. Concerning representational formats, we confirm that the use of tree diagrams allows students to abandon manipulative strategies and move towards generalisation. Furthermore, we have analysed how such diagrams are used to construct the multiplication principle, in some cases extending recent research findings. In particular, we have seen that the diagrams have been used by students at different stages of cognitive processing spontaneously, using them to represent particular cases from which the multiplication principle is derived efficiently without specific instruction.

We highlight the importance of using materials in problem combinatorial problems. As we observe in the analysed cases, students can use them till they feel comfortable with written representation. This has been observed in different stages of the inductive procedure, which lead students to generalise the multiplication principle and justify their conjectures.

This study support that experience with two-dimensional combinatorial problems help students to adopt more efficient strategies for three-dimensional combinatorial problems, as English suggested in previous studies.

The findings have a direct impact on instruction in the area of combinatorics. Although we do not question the importance of instructional programs including the use of tree diagrams to generate algorithms for enumeration and counting, we suggest that they should not be the first approach to combinatorial problems. We base this recommendation not only on the fact that some students are able to produce for themselves this type of representation, but on the fact that some students face basic combinatorial problems using inductive reasoning and not tree diagrams. In these cases, although the process to generate the scheme of the multiplication principle is slower, students advance significantly towards generalisation from particular cases.

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