An exploration of mathematics students' distinguishing between function and arbitrary relation.

Panagiotis Spyrou, Andonis Zagorianakos

Department of Mathematics, University of Athens, Greece

This paper focuses on students' awareness of the distinction between the concepts of function and arbitrary relation. This issue is linked to the discrimination between dependent and independent variables. The research is based on data collected from a sample of students in the Department of Mathematics at the University of Athens. A number of factors were anticipated and confirmed, as follows. Firstly, student difficulties involved vague, obscure or even incorrect beliefs in the asymmetric nature of the variables involved, and the priority of the dependent variable. Secondly, there were some difficulties in distinguishing a function from an arbitrary relation. It was also thought that additional problems occur in the connotations of the Greek word for function, suggesting the need for additional research into different linguistic environments.

Introduction

The concept of function is essential in the understanding and learning of mathematics. It is considered to be the most important concept learnt from kindergarten to college or university (Dubinsky & Harel 1992). The difficulties students experience with this concept can only be understood in relation to its *definition* and the appearing of *cognitive obstacles*. Several researchers found that in the early stages of function teaching in secondary schools that natural models dominate *using mainly 1-1 (one-to-one) functions*. (Evagelidou, Spyrou, Gagatsis, & Elia 2004; Elia & Spyrou 2006).

The reliance on the natural models means that the connection between the dependent and independent variable is emphasized rather than focusing on the priority of dependent to the independent variable. Furthermore, the natural models which are offered to the pupils are idealized, distant from the realities from which they were created and described in analytical formula, thus making it "difficult for the students to distinguish between relationships discovered by experience and the mathematical models of these" (Sierpinska 1992, 32). This approach results in a difficulty in realizing that the dependent variable is a magnitude which is used to estimate a measurement and that the independent variable is the means for this particular purpose, with or without an analytical formula.

The etymology of the Greek word for "function" introduced a note of caution. The root of the Greek word for function ("synartisi") is different from the origin of its Latin equivalent which is mainly *operational*. In colloquial Greek when a person or abstract phenomenon such as time, speed or measurement has a functional relation ("synartate") with another person or abstraction, the effects tend to be symmetrical. A bond is implied, whether active or inert, which is triggered when "one side" (usually either side) is altered, evoking a change in the "other side". Therefore, the common perception of the Greek word for "function" implies the symmetry of the function variables. This symmetry might create a difficulty in understanding the difference between the variables in the mathematical definition, i.e. which is the means and which is the one to measure. Sierpinska (1992) recognized this difficulty as the obstacle, **"regarding the order of variables as irrelevant"** (p. 38). This definition of the obstacle is the starting point of this paper.

To conclude, the over reliance on one-to-one correspondence in function teaching and the common perception of function creates an obstacle which may persist throughout Higher Education. It was therefore decided to research the persistence of this obstacle in the thinking of students in Higher Education.

Theoretical framework

The definition of function went through several stages until it reached its present form. This progressive development has given rise to a number of epistemological obstacles. The first definitions of the concept of function by Bernoulli, Euler and Cauchy were not complete. This is because they saw the symmetricality of the dependent and independent variable, in the context of a relationship. Sierpinska marks this "moment" in the concept's theory as an epistemological obstacle: *EO(f)–5 (Unconscious scheme of thought)* ***"Regarding the order of variables as irrelevant"*** (1992, 38). It was Dirichlet in 1837 that used his study on Fourier series and the conditions under which a Fourier series converges, and formulated a general definition of function: *"if a variable y is so related to a variable x that whenever a numerical value is assigned to x there is a rule according to which a unique value of y is determined, then y is said to be a function of the independent variable x"* (Boyer 1968, 600). Dirichlet's definition of function is still in use and his main conclusion states the necessity of the dependent coordinate being uniquely determined and not always the inverse. Therefore, the Dirichlet definition expressed precisely, for the first time, the notion of a **mediated measure** in the concept of function. That is to say: to estimate the dependent variable y, and to achieve it although there is no immediate access to y, is to measure it through x. Therefore the independent variable is the mediating variable which gives access to the dependent variable, resulting in the priority of the latter.

The literature on the study of the epistemological obstacles that occurred through the development of the definition of function is particularly rich (e.g., Freudenthal, 1983; Sfard, 1992; Dubinsky & Harel, 1992; Sierpinska, 1992; Even and Tirosh, 1995). However, it is difficult to find any research on the particular obstacle which is the subject of this research.

As discussed in the introduction, the teaching of function during the first years of school is oriented towards a common perception of function, emphasizing the relation between the dependent and the independent variable, disregarding the priority of the dependent variable. Furthermore, the dominant use of one-to-one functions makes it harder for students to recognize the importance of the dependent variable, which is the target of the measurement. In addition, the focus on the relational aspect of biunique functions conceals the richness of the applications which the definition allows.

Finally, the common perception for teaching function in school can be seen as a collection of habits, whereby ***"the character of a habit depends on the way in which it can make us act"*** (Peirce 1958, Vol. V, par. 18). It is this *habitual comprehension of function,* diverging from the formal definition, which results in mathematical obstacles and difficulties. Thus, we formed the following *research questions* (accompanied by their relation to the questions of the questionnaire that was given to the students —the justification for which is given in the next section):

[1] *Can the students recognize the difference between the concepts of function and of an arbitrary relation?* (1st, 2nd question of the questionnaire)

[2] *Can the students distinguish the order of the variables x and y, and the asymmetry that they have?* (3rd, 4th question of the questionnaire)

Methodology

The methodology arose from the theoretical framework of this paper and sets out to test the persistence of the mathematical obstacle described above, in students in Higher Education. It was anticipated that students would encounter problems in recognizing the priority of the dependent variable and in distinguishing between a function and arbitrary relation.
Within this theoretical framework it was decided to design a survey in two phases.

Phase 1 included the completion of a 4 questions questionnaire by the students *(see Figure 1 below)*. The format of the questions in the questionnaire included two types of questions:
1] Crosscheck items corresponding to questions: **(key Q=questionnaire)**
A) (Q1) Students were given 3 correspondences on graphs and another 3 with table values. They were asked to find out which represented functions.
B) (Q3) Students were given 4 functions on a graph. They were asked which of the 4 would still be function if the lines on the graph were rotated by 90°.
2] Open-type questions with short answers *such as:*
A) (Q1) Students were asked to make the necessary changes to the graphs and table values to change the arbitrary relations into functions.
B) (Q2) Students were asked to give two examples of (arbitrary) relations which were not functions, one algebraic and one represented graphically.
C) (Q3) Students were asked to justify whether the 4 functions on the graph remained functions when turned 90°.
D) (Q4) Students were asked in which case(s) *the 90° rotation of a function's graphical representation represents a function* and to give a *general rule.*

Figure 1 The Questionnaire given to the students

❶Which of the following relations are function relations? Make the necessary corrections to the rest of them, in order to transform them into functions.

x	y
-1	0
0	1
1	2
2	3
3	4

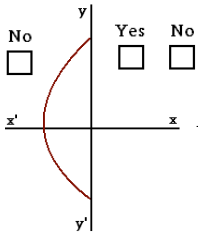
Yes ☐ No ☐

x	y
-3	2
-2	2
-1	2
0	2
1	2

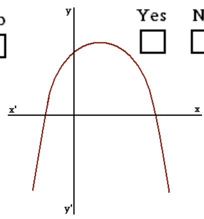
Yes ☐ No ☐

x	y
5	3
-3	2
5	1
0	1
-3	6

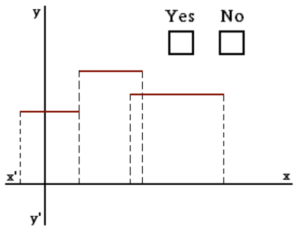
Yes ☐ No ☐



Yes ☐ No ☐



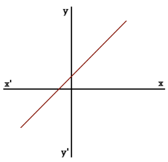
Yes ☐ No ☐



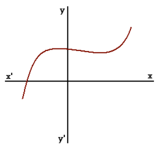
Yes ☐ No ☐

❷Give two examples of arbitrary relations that are not functions (one described graphically and one analytically (with an algebraic formula)).

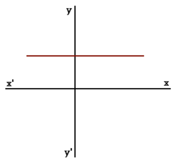
❸What happens to the graphical representations of the following functions if the line on the graph is rotated by 90°? Are they still functions? Give a short justification in your answer.



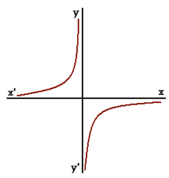
Yes ☐ No ☐



Yes ☐ No ☐



Yes ☐ No ☐



Yes ☐ No ☐

❹In which situation(s) does a 90° turn of a function's graphical representation represents a function? Which general rule would you use?

The questions in the questionnaire were designed to correspond to the research questions posed in the theoretical framework, as follows:
A) The 1st question asks students to distinguish which correspondences are functions and which are arbitrary relations. The 2nd question asks them to give two examples, one graphical

and one with an analytical description, of an arbitrary relation, which does not fit the function definition. The answers to both of these questions correspond to the 1st research question.
B) The 3rd question asks students to distinguish when a function remains a function on a graph when the graphical representation is rotated by 90°. In the 4th question they were asked to interpret this movement and to justify their answers with reference to the application of the general rule. This 90° turn on the graph occurs as a consequence of the permutation between the dependent and independent variables. Therefore, the students are tested in their ability to realize the asymmetry of the x and y variables, i.e., the 2nd research question.

The questionnaires were distributed to 17 students who attended lessons in "Epistemology of Mathematics", in April 2009, in the Mathematics Department of the University of Athens. The course was chosen because of the accessibility to a wide range of students on different courses, theoretical and applied, and at different stages of study. Student participation was voluntary; the questions were answered without time constraints, taking them approximately half an hour to complete.

Phase 2 of the research took place after about 3 weeks (May 2009) and included semi-structured interviews, with 13 students out of the 17 who participated in the questionnaire. Students were fully informed of the objectives of the research and gave their permission for their interviews to be recorded. In all interviews the interviewer emphasized that the purpose of the interview was not to examine the students but was to explore what they thought when they answered the questionnaire, regardless of whether the answers were right or wrong.

Each interview aimed to clarify the answers given on the questionnaire and the problems the students had encountered completing it. Students were asked additional questions in order to start a general discussion and explore their understanding of function with regard to their school and university education. They were asked about (a) their experience of the concept of function at secondary school, (b) their acquisition of the concept of function at the university, and (c) what they thought the use of function was outside the mathematical frame.

All the interviews were audio-recorded and listened to, in order to assess them against the research questions. This assessment showed a consistency in the interview results. Four representative interviews were selected, using the following criteria:
(i) Variety of students' responses to the research questions as shown in both the questionnaire and the interview.
(ii) The interviews comprised different levels of understanding: 1 high (Georgia), 1 moderate (Diana), 1 low (Iris). The 4th interview highlighted the findings of the research (Thanos). The most relevant parts of the interviews for the research were transcribed. These were then divided into 5 minute intervals or divided according to the research questions, and accompanied with short comments. Due to the limitations of space this paper contains just a few but characteristic dialogues from the interviews.

Description of the results — Discussion

Summarizing the main findings of our research, we observed the following:
R1) All students interviewed, except one, gave only examples of 1-1 (biunique) functions. They said they had used examples they recalled from learning functions at school.
R2) All students interviewed, except one, had difficulties in giving good explanation, or any explanation, for the "many-to-one" condition in the definition.
R3) The students used ***mnemonic rules:***

A) Seven students out of 17 (41%) gave the same example (the circle example) as a graphical representation of an *arbitrary relation* that is not a *function* (Q2)**.** Moreover, all the interviewed students except one used inclusively 1-1 examples of functions, an attitude that shows *the strong connection that the students still have with their early function experience*. This connection was admitted extensively in the interviews.

B) Extract from Diana's interview, typical of the definitions used as mnemonic rules and the confusion that follows:

> D: I try to recall the definitions, as far as I can. When I cannot, I "put my mind to work".
>
> R: Does your "mind" ... agree with the definitions?
>
> D: On this occasion it agrees, not always.

R4) 29,5% of the students (5/17) gave the wrong answers to the 1st as well as the 2nd question of the questionnaire, where they were asked to recognize the difference between the concepts of function and arbitrary relation.
R5) 47% of the students (8/17) gave the wrong answer to the 3rd question of the questionnaire, revealing a difficulty in distinguishing the order of the variables x and y, and their asymmetry.
R6) 64,7% of the students (11/17) gave the wrong answers to the 4th question of the questionnaire, revealing a difficulty in distinguishing the order of the variables x and y, and their asymmetry.
R7) Most of the students interviewed have only been concerned with functions in the context of their school and university education. Nevertheless, their education did not equip them with the necessary tools for interpreting the concept of function.
Iris is a typical example: although she reported that *she always thinks of graphical representations when she deals with functions,* she still had difficulties in giving examples of arbitrary relations which are not functions. She did not understand what a 90º rotation of the graph meant despite knowing the formal definition of a function and applying it correctly in the 1st question.
R8) The interview with Thanos is indicative of the students' confusion with the "many-to-one" condition of the definition. For instance, concerning the use of functions, he reported as follows:

> "Wherever I want to put factors say, the x and y are essentially factors. (For example), x is able to measure temperatures and y to count days. Or x to count children and y to estimate the tax return. That is, apart from the fact that we put it in a two-dimensional frame and take a mathematical perspective; the two dimensions (the 2-axes coordinate system) are essentially two parameters. The three dimensions are three parameters, and so on."

It is apparent that he has misunderstandings concerning the priority of the dependent variable. He also considers $\mathbf{x_1, x_2, x_3}$ as parameters of the function $\mathbf{f(x_1,x_2,x_3)}$, which "lead" to the y variable, thus showing that he is confusing the function of many variables with the "many-to-one" condition of the definition.

The results from the questionnaires and interviews confirmed the problematic areas anticipated at the outset of the research. It is evident from the results of the questionnaires and the interviews that students experienced difficulties in answering all four questions. The most difficult question is question 4 (R6) where 2 out of every 3 students experienced difficulties. When incorrect answers for question 4 were combined with the incorrect answers for question 3, where almost half were wrong (*R3*), they provided evidence of *the difficulties students have in distinguishing the order of variables x and y, and their asymmetry.*

Although there is a smaller percentage (29.5%) of wrong answers for the 1st as well as the 2nd question (R4) their weakness indicates the persistence of the problem. They show difficulties students have in *recognizing the difference between the concepts of function and of arbitrary relations.*

The results from the interviews (see R1, R2, R3, R7 and R8) provided data that revealed more than the questionnaires. The majority of the students had separate ideas about the definition and its application. All of the students knew a formal definition. Only 2 students gave examples of "a single-valued but not uniquely invertible function". Many students still experienced difficulties when asked about this specifically in their interview. The dominance of the biunique functions is further evidence of *the dominant influence of the first years of function teaching.*

From the evaluation of the questionnaire data we conclude that there are difficulties in students' abilities to recognize the difference between the concepts of function and of arbitrary relation. However, most difficulties occur in students' abilities to distinguish the order of the variables x and y and their asymmetry, confirming Sierpinska's (2002) claim (EO(f) – 5).

In our opinion, the etymology of the Greek word has an additional negative impact in students' ability to overcome the mathematical obstacle, by encouraging a common perception in favour of the biunique function. This common perception is handed down through a school's teaching methods, often remaining unchallenged. We think it would be worth testing the same mathematical obstacle in different environments, to isolate the influence of the etymology of the Greek word and its' part in the persistence of the obstacle.

References

Peirce, C. S. 1958. Collected papers of Charles Sanders Peirce, Volumes I-VI (C. Hartshorne & P. Weiss, Eds.). Cambridge: Harvard University Press.

Boyer, C. 1968. A history of mathematics. New York: Wiley & Sons.

Freudenthal, H. 1983. The didactical phenomenology of mathematical structures. Dordrecht: Reidel.

Dubinsky, E., and G. Harel. 1992. The nature of the process conception of function. In E. Dubinsky and G. Harel (Eds.), The concept of function: Aspects of epistemology and pedagogy 85-106. Washington, D.C.: Mathematical Association of America.

Sfard, A. 1992. Operational origins of mathematical objects and the quandary of reification—The case of function. In E. Dubinsky & G. Harel (Eds.), The concept of function, Aspects of epistemology and pedagogy 59-84. Washington D.C.: The Mathematical Association of America.

Sierpinska, A. 1992. On understanding the notion of function. In E. Dubinsky & G. Harel (Eds.), The concept of function, Aspects of epistemology and pedagogy (pp. 25-28). Washington D.C.: The Mathematical Association of America.

Even, R., and D. Tirosh. 1995. *Subject-matter knowledge and knowledge about students as sources of teacher presentations and subject matter.* Educational Studies in Mathematics 29: 1-20

Evagelidou, A., P. Spyrou, A. Gagatsis, and I. Elia. 2004. *University students' conception of function.* In M. J. Høines and A. B. Fuglestad (Eds.), Proceedings of the 28th Conference of the International Group for the Psychology of Mathematics Education 2: 251-258. Bergen University College, Norway.

Elia, I., and P. Spyrou. 2006. *How students conceive function: A triarchic conceptual-semiotic model of the understanding of a complex construct.* The Montana Mathematics Enthusiast 3(2): 256 -272