# The role of proof validation in students' mathematical learning

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The study of proofs is a major obstacle in the transition from school mathematics to university mathematics. Given the importance of argumentation and proof in the spectrum of mathematical activities, the incoming students' understanding, appreciation and knowledge of the nature and role of proof must be considered. I describe the results of an exploratory study of first year mathematics undergraduates' criteria and learning process when validating mathematical arguments or proofs. The study is based on a series of written tasks and interviews conducted with first year honours mathematics students at NUI Galway. I presented the whole class with numerous proposed proofs of mathematical statements, and asked them to evaluate and criticize those. The first year students' written comments on different and partly incorrect 'proofs' of mathematical statements revealed some information about their criteria when validating mathematical arguments. In recently held interviews with eight randomly chosen students I focussed on the learning experience during the process of proof validation. Considering the observed learning effect and its large potential extent during the process of proof validation I propose its practice in the teaching of mathematics.

**Keywords: proof, proof validation, transition to University mathematics**

In the first part of this article I explain what is meant by *proof validation* and consider various aspects of this activity, including how and why mathematicians validate mathematical arguments. I will then describe our experiments and findings, and consider the undergraduates' validation skills and practices in relation to previously characterized aspects of proof validation. I emphasize the learning effect during the process of proof validation and finally argue for explicit inclusion of its practice in the teaching of mathematics, as the development of validation skills not only improves the practice of validation itself, but also the ability to construct proofs, the understanding of mathematical context, the knowledge of proving strategies and the links between different areas of mathematics.

## On proof validation.

Before considering how students validate proofs I discuss the nature of proof validation; I further distinguish it from other types of reading and from construction of proof. Selden and Selden (1995) call the readings to determine the correctness of mathematical proofs and the mental processes associated with them *"validations of proof"*. I extend this description of proof validation considering that mathematicians, advanced students or maths teachers validate not only to determine the correctness of an argument. Discussions with a number of experienced mathematicians suggest that they also wish to reach an understanding of why a mathematical statement is true and often to understand the content and the position of the proved statement in a wider context. Factors that experienced mathematicians might consider while validating proposed proofs include: whether the argument provides the reader with understanding,transparency and quality of proof idea and strategy, clarity of the structure, whether the reasoning is precise, correct and sufficient, and whether the argument is convincing. In Section 2 below I report on my investigation of the criteria that students consider as essential for a valuable mathematical argument.

***The activity 'proof validation'.***

Validation, in comparison to the reading of non-mathematical texts, requires the reader to put some additional effort into understanding of the reasoning. Validation usually takes more time, the validator might consider the whole proof or parts of it several times and might be more inclined to write a few notes checking deductions, verifying justifications, etc. According to Selden and Selden (2003) the mental process when validating proofs can include for example asking/answering questions, constructing subproofs or recalling other theorems and definitions. It is well documented that construction of proof is a major obstacle for students. Selden and Selden (2003) describe how the ability to validate proofs relates to the ability to construct them. On the one hand proof construction and proof validation are **different.** Proof construction requires 'the right idea' at the 'right time'. The validation process can usually be managed in a linear order, unlike construction of proof. On the other hand proof construction and proof validation **entail each other** as one considers during the process of proof construction how that proof would be validated, and as validation of a proof is likely to require the construction of **subproofs**. I summarize this relationship in the following diagram.

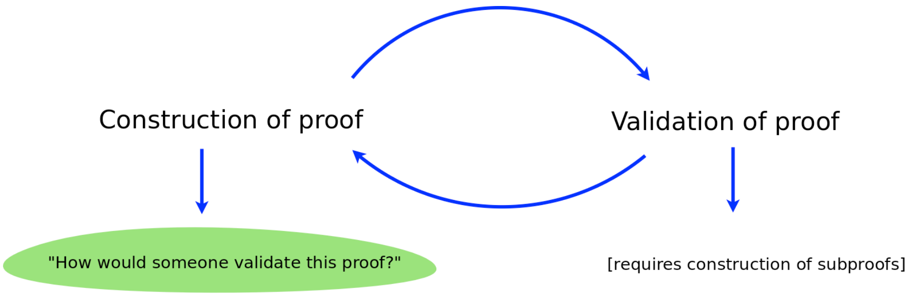


Figure 1: Construction related to validation of proof

Considering my comments above on the nature of proof validation, I extend this diagram to highlight the extent of the learning effect through the process of proof validation.

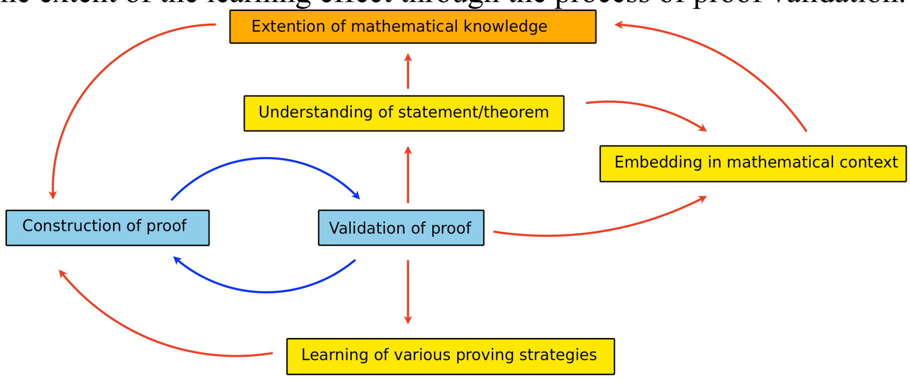


Figure 2: Proof Validation in the process of learning about mathematical proof

I summarize my **hypothesis**: *The ability to validate proofs can improve the ability to construct proofs, develop deeper understanding of the meaning and significance of the proved theorem and develop knowledge of proving methods or strategies.*

## The experiment.

The study is based on several tests and interviews conducted with first year honours mathematics students at NUI Galway. The students' way to validate mathematical proofs first caught my interest when I was analysing their responses to a written exercise that I held in May 2008, at the end of their first year at University. My research has focussed on that topic since. Therefore the design of further research instruments concentrated on proof validation. The test for the new incoming students (D-test08), held in September 2008 with 103 participants, included tasks designed to give me some insights into their proof validation skills. Based on the findings of the analysis of the written exercises I designed interviews to be held with a smaller number of students. The aim of these oral exercises was to get a deeper insight in the students' validation processes. The interviews were held with eight undergraduates in March 2009. I report below on findings arising to date from all three of these exercises. Analysis of the data is ongoing. Some of the questions had been adapted (with permission) from the Longitudinal Proof Project which ran 1999 until 2003 in the U.K. Considering the previously described aspects of the process of proof validations I investigate how our first year students validate mathematical proofs, which criteria they use to decide whether an argument is correct or not and in the learning process during the exercise of validating and comparing different mathematical arguments.

### *Written exercises.*

The test held in May 2008 was attended by 37 participants. The students were presented with six attempts to determine, with proof, whether the statement

> ***"When you add any two even numbers, your answer is always even"***

is true or false. For each of the attempts, I asked the students to give a mark out of five and a line of advice.

### *Observations from written exercises: students' criteria when validating proofs.*

The prevalence of some expressions indicates what the students found essential in a good proof. I list below a number of themes that caught my interest, either because they appeared quite often or because they surprised me.

*The role of examples.*

When confronted with a few examples to show the truth of the statement, I found that most students recognized the necessity of rigour and commented on that. *"He has just given examples, this is not a proof"* or *"Not a general solution"* are typical remarks. Some of the students' comments indicate that *"proof by example"* might be seen as another type of proof, just not as good as one including a general formula: *"It's not bad. But it's only proved by example."*, *"Although this does prove the statement, it only does so for a few egs..."*. On the other hand, students do give examples a surprisingly high if not essential value within a proof. They often deduct marks on the basis of absence of examples. Some students comment on the absence of examples to explain the answers (*"Give an example to show ..."*) but most criticize the lack of examples as an essential part of the reasoning. Some students request examples in (correct) answers to *"back up proof"* or to *"finish the proof"*. One student criticizes a correct answer because the statement is *"not proven by numerical example"*. Without any examples students don't seem to be satisfied by an argumentation.

*"Not mathematical enough"*

The following are a few selected comments on a correct answer that was expressed purely in the form of text: *"The proof makes sense but she could have used a more mathematical approach"*, *"Good intuitive answer but needs a mathematical proof"*, *"Correct answer but show mathematically"*, *"The proof should be shown mathematically as well as in words"*. The argument formulated as text was *"not mathematical enough"* to most of the undergraduates. In comparison to another more algebraic looking approach one student comments: *"Although Cathy's answer is true there is too much English and does not mathematically prove it unlike Aoife"*. These comments raise the question of what the students associate with the term *"mathematical"*. Their comments indicate that *"mathematical"* appears to mean including formulas (*"Try to come up with formula"),* algebraic equations (*"Give clear equation to support your answer", "Would like to see this expanded with a general equation"*) and mathematical notation (*"Use mathematical notation to show this", "Cathy's answer is well written and ... although she should sum her answer up .. using formal notation"*). Consequently another answer, which is fundamentally and irreparably incorrect but includes algebraic equations and mathematical notation, seemed basically correct to more than half of the students. Fewer than 30% of the students recognized that this answer was wrong.

A visual approach to prove the statement is generally not accepted by the students. The proposed answer consisted of a diagram showing how an even number can be represented by two rows of dots, and how addition of even numbers can be interpreted as concatenation of two such representations. Most students interpreted the answer as just one example, visualized in a diagram. *"Again Finn's answer only covers 1 solution. He needs to give a general statement."*, *"This proves that it works for 12+8. It doesn't prove for all cases."*, *"Not a proof, just an example"* are a few typical answers. 27% recognized the idea behind the illustration, but didn't acknowledge a graphical representation of the correct idea as a mathematical proof: *"Good visual proof but use mathematics"*, *"Good visual representation but needs notational explanation"*. A proof without numbers and words can't be sufficient: *"There are no words in this proof"*, *"Proof is illustrated using graphics rather than numbers", "This does not prove anything, words and numbers are needed"*, *"There are no words in this proof"*.

*"it doesn't explain..."*

Some mathematical educators argue that whether or not an argument is accepted as a proof depends not only on its logical structure, but also on how convincing the argument is. That aspect seems to play a role in the students' proof evaluation as well. *"Nice pictures, you could have written a line explaining it though", "Intuitively correct but needs to explain why the answer means the statement is true", "Need more explanation"* or *"She should explain what she is doing"*. The positive comments on the highly marked answers often include a note about the good explanations: *"Aoife has a very clear and straightforward answer"*, *"Well explained answer"* or *"Aoife is using clear and simple language to get her answer across..."* are a few comments on the students' favourite answer. Those comments indicate that just having a good idea to prove a statement is not sufficient for the students. The skills of convincing and explaining ideas to others matter to them.

I summarize that after their first year in university most students are aware that checking the truth of a statement for a few examples is not sufficient to prove the statement. On the other hand examples play an important role in mathematical argumentation to students. Even after accepting the correctness of an argument they are not convinced until it is shown with a few examples. Overall the students' picture of proof seems to be vague. To them a valuable proof should have a certain structure, starting with a definition, followed by some algebraic equations or formulas, and finishing with some examples. Structure and formalism seem to be more important to the students than the idea behind the proof. If these requirements are met most of the students give at least a few marks regardless of the correctness of the particular steps or whether the overall idea makes sense to them or not. It seems, a good idea to prove a statement is not being valued as highly as the structure and formalism of a proof. Beside structure and formalism the quality of explanations played a role in the students' proof evaluations. It seemed that if an answer didn't show attempts to convince the reader of an argument, most of the students would deduct marks.

***Oral exercises.***

In March 2009 I held interviews with eight randomly chosen students who had attended the written exercise in September as well. The aim was to get a deeper insight into

- students' opinions about valuable proofs. What do students mean when they use the term “mathematical”? Do they ask for examples in order to understand the given reasoning or because they consider them as essential part of proofs?
- students' validation process. How do they attempt the task of validation? For example, do they read the proposed proofs line by line? Do they write notes, verify the arguments?
- the learning effect during the validation process.

To facilitate comparison of the results one of the statements chosen for consideration in the interviews was similar to the one in the written exercise, but a bit more difficult.

**The squares of even numbers are even, and the squares of odd numbers are odd**.

The other statement was different from the exercises the students had performed in the context of this study so far.

**Let *f* be a quadratic function, $f(x) = ax^2 + bx + c$ with $a,b,c \in R$ and $a>0$.**

**Show: *f* can't have more than two common values with its derivative $f'$.**

Again the students were confronted with five or six different arguments, some incorrect or partly incorrect, and asked to comment on them and finally rank them. Some of the proposed proofs were algebraic, some visual, some written in text, others wrong but expressed using “typical” mathematical notation.

*Observations from the oral exercises: the learning process when validating and comparing different mathematical arguments.*

A detailed analysis of the interviews is in progress. In structuring and partly transcribing them a remarkable pattern caught my attention. The students were very quick in deciding whether they liked an answer or not and their first opinions and comments were similar to those in the written exercises. During the meeting though I could observe a process of understanding when spending more time with the task, comparing ideas with some proofs they have seen somewhere else, and sometimes even questioning their own criteria. Especially when ranking the different answers the students reconsidered their opinions and sometimes changed their minds about certain answers. These findings are in line with Selden/Selden and Alcock/Weber:

- Selden and Selden describe that students' performance in distinguishing valid from invalid arguments improved dramatically through the reflection and reconsideration during the interview. (Selden and Selden, 2003)
- “Our results suggest that many of the students in our study could perform this task competently, but did not do so without prompting.” (Alcock and Weber, 2004)

The fact that the students were forced by the ranking task to spend some time on their reflections encouraged a learning process. I conclude that the understanding of mathematical concepts can improve considerably during the process of careful proof validation.

## Conclusion

The findings show on the one hand that undergraduates have a vague yet inflexible picture of valid proofs. Structure and 'mathematical' looking formalism seem more important to them than the idea behind its appearance. On the other hand I discovered that reflection during the process of proof validation encourages a learning process about the nature of mathematical proofs. Recalling the discussion in the first part of this article about the nature of proof validation, in particular the relation between construction and validation of mathematical proof and the attainment of understanding through validation, I conclude that practice of proof validation can not only improve students' validation skills but can also lead them to a better understanding of mathematical content and to improved appreciation of deductive reasoning.

## References

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