Aspects of a teacher's mathematical knowledge in a lesson on fractions

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This paper is about a mathematics teacher, and how aspects of his mathematical knowledge surfaced in a 5th grade (11 years old) fractions lesson in Norway. The teacher's responses to pupils' (unexpected) comments and questions, 'contingent moments', are discussed. Difficulties in dealing with improper fractions, which were mirrored in the pupils' inputs in the lesson, are discussed. Considerations are made whether the problems the pupils expressed can be traced back to aspects of the teacher's mathematical knowledge.

Keywords: Improper fractions, teacher knowledge, contingent moments

Background and introduction

What knowledge is required for the teaching of mathematics has been widely discussed within mathematics educational research, both with regard to what comprises the knowledge, and how this mathematical knowledge is made accessible to others. Through classroom observations and focus-group meetings with four mathematics teachers in 5th grade elementary school in Norway I have been studying how teachers drew on their knowledge in mathematics and mathematical didactics in their teaching. Lessons were videotaped and I have used the Knowledge Quartet developed by Rowland, Huckstep and Thwaites (2005) as an analytical framework to study how a teacher's (Hans') mathematical knowledge surfaced in the lesson. How examples and illustrations of improper fractions influence the pupils' conceptions and difficulties are discussed. Before presenting the lesson with the teacher Hans, I will report some research about mathematical knowledge for teaching which I have used in my study, and also briefly report research about fractions which suggests some factors explaining why pupils' concepts of fractions only become partly developed.

Mathematical knowledge for teaching

By questioning how teachers' use their knowledge in the subject they teach and where teachers' explanations come from, Shulman (1986) brought didactics into mathematics educational research. Shulman suggested distinguishing among three categories of content knowledge: Subject Matter Content Knowledge, Pedagogical Content Knowledge and Curricular Knowledge. *Subject Matter Content Knowledge* (SMK) refers to the knowledge the teacher has in mind, both substantive and syntactic. *Pedagogical Content Knowledge* (PCK) goes beyond knowledge of the subject and refers to content knowledge for teaching, "It is the capacity of a teacher to transform the content knowledge he or she possesses into forms that are pedagogical powerful" (Shulman, 1987, p.15). *Curricular Knowledge* is both lateral and vertical. Lateral curriculum knowledge is how the teacher is able to relate the content and issues discussed in his/her subject to that being discussed in other subjects. Vertical curriculum knowledge is about what has been taught in earlier lessons (and years) within a subject as well as what is relevant to be taught in the next lessons.

Rowland et al (2005) based their work on Shulman's categories of knowledge. Through a grounded approach to data from video studies, the Knowledge Quartet (KQ) was identified. In the KQ the classification of the situations in which mathematical knowledge surfaces in teaching is of importance (Rowland and Turner, 2008).

The Knowledge Quartet has four broad dimensions; Foundation, Transformation, Connection and Contingency. *Foundation* is the mathematical knowledge the teacher has gained through his/her own education, it is knowledge possessed and which can inform pedagogical choices and strategies. It is the reservoir of pedagogical content knowledge you draw from in planning and carrying out a lesson and thus informs pedagogical choices and strategies. *Transformation* focuses on the teacher's capacity to transform his or her foundational knowledge into forms which can help someone else to learn it. It is about examples and representations the teacher chooses to use. The third category, *Connection*, binds together distinct parts of the mathematics and concerns the coherence in the teacher's planning of lessons and teaching over time and also coherence across single lessons. *Contingency* is the category which concerns situations in mathematics classrooms that are impossible for the teacher to plan for; the teacher's ability to deviate from what s/he had planned and the teacher's readiness to respond to pupils' ideas are important classroom events within this category.

Fractions

As with decimals and percentages fractions occur with different meanings. These meanings can also be seen in everyday life. A fraction can be a part of a whole, a place on the number line, an answer to a division calculation or a way of comparing two sets or measures (part group). Novillis (1976) studied the hierarchical development of various aspects of fractions among American children. She found that the part-whole and part-group models were significantly easier for the children to understand than the number line. Her study referred to work with fractions not bigger than one. As opposed to the part whole or part group model, the number line does not incorporate that a fraction can be thought of as a concrete object. But according to Dickson, Brown and Gibson (1984), a number line makes improper fractions appear more natural. They claimed that "the representation of fractions as sub areas of a unit area does not lend itself very well to the representation of improper fractions" and that "The acceptance of the definition of a fraction as meaning 'part of a whole' is inconsistent with the very existence of such improper fractions" (p. 279).
According to Anghileri (2000) much of the focus when working with fractions in school is identification of fractions as part of a whole. She claimed that success in working with fractions depends on the ability to see the fraction both as representing a number and a ratio which reflects the procedure for finding the number. She wrote:

> Research suggests that an approach to fractions which identifies each as numbers to be located on a number line, without emphasizing the way of partitioning a whole, will help to establish the equivalence with decimals and percentages (p.115)

She thus warned against emphasizing fractions as parts of a whole in schools. This is in accordance with Askew (2000) who claimed that if one focuses on fractions as part of a whole so that becomes a social convention, possibilities for a obtaining a well developed fraction concept are limited.

Hans

Based on an analysis of a lesson with Hans using the KQ as an analytical framework, I will discuss how aspects of Hans' mathematical knowledge became visible in this lesson, and if difficulties in dealing with fractions bigger than one whole can be traced back to aspects of the teaching. I will present an account of the lesson before I analyze it in terms of the aspects of the KQ, followed by a discussion.

The lesson objective was written on the smart board when the lesson started: 'To be able to calculate with fractions which are bigger than one whole'. Hans asked the class what the objective involved and a pupil suggested 4/3 or 1 1/3. The first task was 6/8+5/8 and it was illustrated by two rectangles on the smart board. A pupil was taken to the board. He worked it out and used the smart board to illustrate how one rectangle was filled up to a whole and three remained in the other.

The next task was illustrated by circles divided into eight pieces. There were two circles on each side of an equal sign. Five pieces in the first and four in the other on each side were shaded. To Hans' question how much it was all together Jens first suggested 11/9. When the teacher did not confirm, Jens suggested 10/9. Hans then asked how much was shaded in the first circle, and Jens said 5/9. 'Are there nine?' Hans then asked, and he let a pupil in class answer 8. What about the next circle? Hilde suggested ½. Hans confirmed but converted the half into 4/8 'for the sake of clarity' he said. The answer 9/8 was agreed upon, and a pupil came to the board converting it into 1 1/8. A girl, Petra, then demonstrated uncertainty (misconception) asking: 'How is it possible to take nine eighths? When there are eight bits and then take one more? How is that possible?' Pupils in class explained.

For the next two examples, Hans chose not to use illustrations; 'let us try without', he said. The tasks to be done were: 3/5+3/5=, and 7/10+5/10=. For the last one a pupil suggested converting 12/10 into 1 1/10. After having clarified that it became 1 2/10, Hans wanted to go on to the next task, when Mads had his hand up suggesting converting 1 2/10 into 1 1/5 'like the first one', he said.

The last task was to take away ¼ from 2. Hans had drawn 2 circles on the smart board, and he wrote 2-1/4 under the circles. The circles were divided into four pieces which all were shaded. Espen was taken to the board to work it out. He said: 'it is two minus one fourth' and he erased two ¼ pieces of the second circle. Mads then shouted: 'Now you are erasing one half', followed by: 'not one fourth of two'. Hans asked Espen if he was sure and how much he actually should have taken away and Espen answered: 'If it is two so half of that' (pointing to one of the circles). Hans emphasized 'what is the whole'. Then Espen realized that he had not done what the teacher had expected. He shaded one of the quarters he had erased and wrote 1 ¾ =7/8. Then Hans asked him to explain what he had been thinking.

After this episode Ella went to the board and pointed to the digit 1 in 1 ¾ and asked: 'what does the *one* actually do there?' Hans responded to her question by taking it all from the beginning, asking questions for Ella to answer throughout his review. He started by asking what the whole was and how many pieces they originally had. Four more times during Hans' review, Ella repeated her question. When he had finished and Ella expressed not understanding, Hans erased the board and tried again. This time he just focused on 4/4 being one whole. Then Ella said she understood.

Foundation

What subject knowledge (SMK) did Hans have on which he could draw in this lesson? In the textbook fractions were illustrated as part of rectangles, as part of circles and on number line. Hans used rectangles and circles, but not the number line. He demonstrated that he knew how to make fraction tasks adding up to more than one. Hans 'converted' ½ into 4/8 'for the sake of clarity'. He foresaw that writing ½ could cause confusion and wanted to avoid the issue of common denominator at this stage. He also inserted an equal sign when that was missing in Espen's work with the subtraction task on the board.

Transformation

How was Hans' foundational knowledge set out in practice? The lesson seemed to be well planned. He had prepared examples and illustrations in form of rectangles and circles. The smart board was used to illustrate how it was possible to fill up one whole and then 'see' what fraction was left. There was a dialogic approach, Hans invited the pupils to participate and he let them come to the board to work out the exercises he had chosen. He also let other pupils explain when errors and misconceptions surfaced. Thus the pupils took actively part in the lesson.

Hans' choice of examples and illustrations mirror a view on fractions as part of a whole. He did not use the number line. Neither did he use examples which mirrored fractions as part of a group or proportions.

In the first example Hans used the expression 'eleven out of eight' which may have caused the question from a pupil about how it was possible to take nine eighths, when there were eight bits and then take one more was grounded in that expression.

All exercises included converting an improper fraction into a mixed number. In all but the first example (which was taken from the text book), the fraction part of the mixed number was a unit fraction. This suggests why a pupil converted 12/10 into 1 1/10 which again reveals that she had an undeveloped conception about the link between an improper fraction and a mixed number. The reason why the tasks Hans made all had a unit fraction as the one remained is not clear. One suggestion is that Hans only thought about making the answer bigger than one whole, and one bigger was sufficient. Another suggestion is that he wanted to avoid abbreviations since that was not the goal of the lesson.

With regard to the illustration of 2-1/4, it seemed that it was the illustration itself that caused Espen the difficulties. I suggest that Espen would have calculated 2-1/4 without any difficulty if it had not been for the circles being used to illustrate.

Connection

There seemed to be a logical coherence in this lesson. There were good links across the lesson with regard to progression and examples and illustrations. Hans started by asking what the goal meant, and went from there to adding two fractions (6/8+5/8) which he had illustrated with rectangles. Then he went on to another addition with fractions (5/8+4/8), illustrated with circles. For the next two tasks (3/5+3/5 and 7/10+5/10), he suggested 'trying with out illustrations'. Then he went on to subtraction and chose to illustrate with circles again. This task was not only different from the others in terms of not being addition. Now he started from a whole number, 2, taking away a unit fraction (1/4). He also chose to illustrate with circles. There may seem to be a gap between the addition tasks and the subtraction task. The subtraction task was not only another calculation it also involved an integer which none of the addition tasks did.

Contingency

There were four contingent moments in this lesson on which I will comment. First, Jens's answer to Hans' question 'how much is it all together' when having shaded five in one and four in the other. Why did Jens answer ninths, and how did Hans respond to it? I suggest that Jens saw the nine shaded pieces, five in the first and four in the second circle as a whole, and that was why he answered 5/9 when Hans broke the task down asking how many are shaded in the first. From Jens' point of view 5/9 was correct. It did not seem that Hans understood how Jens was thinking. He responded to Jens as if Jens had done an unintended error (counted

9 pieces in the circle instead of 8). Hans did not invite Jens to explain why he kept saying ninths. He let another pupil say eight, and he never came back to Jens and his answers. In this case there was a lack of compatibility between the teacher's and the pupil's thinking. The teacher responded to this contingent moment without incorporating it further into the lesson.

Second, I will comment on Petra's input when she asked how it was possible to become nine eighths; when there were eight bits and then take one more. That the question was unexpected to Hans and that he was not prepared to answer it, were mirrored in how he responded to it. He first repeated the question, then he stumbled before Mads inserted 'you have more pizzas you know', Hans confirmed saying yes, and when Petra again asked: 'One pizza, and there are eight pieces of that', Mads again emphasised his view by saying 'and then you take a new pizza'. In this case, Hans acknowledged Petra's question, but left it for another pupil to explain. An unanswered question is if he did that on purpose, or if the pupil's input helped him out of a problem which he was not able to answer right on his feet.

The third contingent moment in this lesson which I will discuss is Espen's first response to 2- ¼. Obviously, the way Espen first worked out the task was due to that he perceived 2 as the whole and he took ¼ away from that. He took away ¼ of 2 which is ½. Also here, as in the case with Jens, there was a lack of compatibility between the teacher's and the pupil's thinking. However, this time the teacher acknowledged the pupil's contribution and challenged him on how he had been thinking. Hans demonstrated an open way of asking the child, acknowledging his thinking and incorporated it into the lesson.

In the last contingent moment on which I will comment, Hans acknowledged Ella's question why the 1 was there in 1 ¾. However, he did not incorporate Ella's thinking in his answer. He worked out the task over again posing questions and funnelling Ella through *his* doing. Also when Ella repeated her question twice ('but why is the one number there?') Hans carried on with his explanation. When Ella still did not understand, she asked 'but why can it not just be deleted then?' Like in the case with Jens, Hans broke the task down, concentrated upon one of the circles and 4/4 in that being the same as one whole. Then Ella expressed her understanding.

Discussion

Hans responded differently to the contingent moments discussed above. I suggest that the reason why he responded differently to different pupils was based on his knowledge about these pupils' different mathematical abilities. As with Espen, who was one of the best pupils in class, Hans knew that Espen neither was making an occasional error, nor that he did not understand. Therefore he challenged Espen. But did Hans understand *why* Espen had done what he did?

In Hans' lessons there was a confident atmosphere. Due to the atmosphere, the pupils were actively participating by asking questions and commenting on what their classmates were doing on the board. This confident atmosphere was created by the teacher through the way he invited the children's contributions. Hans' knowledge about the pupils' different abilities and the confident atmosphere of his lessons are not aspects of a teacher's knowledge as part of the knowledge quartet. However, analysing a lesson through the use of the knowledge quartet made these aspects of a teacher's knowledge visible and has contributed to draw an even broader picture of how different aspects of a teacher's knowledge surfaced in this lesson.

Finally, I want to discuss the contingent moments which I suggest mirrored some common difficulties the children had in dealing with improper fractions. I have suggested that Espen's way of interpreting the task was grounded in the way it was represented with circles. Although the teacher kept reminding the pupils what the whole was, pupils seemed to have difficulties with it. Jens looked upon the shaded pieces as the whole. Petra did not understand how it was possible to take nine bits when eight was the whole. And Ella argued why the whole had to be there. This suggests a shortcoming with regard to the foundation and transition aspects of Hans' knowledge which incorporates research about what factors that have shown to be significant with regard to pupils' understanding or lack of such of fractions bigger than one. He chose to use the same illustrations of improper fractions and mixed numbers as he had done when dealing with fractions smaller than one, always emphasising that fractions are parts of a whole. In this matter the illustrations he used seemed to cause more difficulties than help in pupils' work with fractions bigger than one. This suggests that emphasising fractions as part of a whole and illustrating fractions as such, can explain some of the difficulties revealed in this lesson.

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