

Some initial findings from a study of children's understanding of the Order of Operations

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This paper presents some of the initial findings of a study into the strategies used by children to solve arithmetic and algebraic problems requiring the appropriate use of the order of arithmetic operations. The research has utilised graphics calculators which have been programmed with Key Recorder Software as a data collection tool. This has enabled the researchers to analyse the children's approaches to some of the questions posed by observing their calculator keystrokes. Interviews with both teachers and pupils will be used to link the pupils' strategies with the teaching methods used, and an initial analysis of observed misconceptions has been carried out. Initially this study has involved children in the UK and in Japan, where teaching methods differ substantially.

Introduction

The principle of the Order of Operations is a cornerstone of the understanding of arithmetic. It is necessary in order to correctly perform arithmetic calculations and it is also an essential prerequisite to the beginnings of the understanding of algebraic structure and the ability to understand and apply the principles of algebraic convention correctly.

More fundamentally, it could be argued that an acknowledgement of the need for a convention in arithmetic is an important step in the development of an appreciation of mathematical convention in other areas of mathematics and indeed to the sense of learning the language of mathematics, where standard rules are necessary in order to assist in clear communication.

In considering the ways in which algebra is developed in different countries, the relationship between algebra and arithmetic is usually characterised by the definition of algebra as generalised arithmetic. Thus the view of 'arithmetic then algebra' dominates school curricula in most countries. The reason for this, according to Lins and Kaput (2004) *can be found in the strong dominance of Piagetian constructivism. As algebra would require formal thinking, while arithmetic would not, and as formal thinking would correspond to a later developmental stage, algebra should come later than arithmetic.* (page 50) This is seen in the work of Kuchemann in Hart (ed) (1981) for the Concepts in Secondary Mathematics and Science (CSMS) project who combined the view of algebra as generalised arithmetic with the Piagetian developmental view. Lins and Kaput (2004) argue that *the most visible result of Kuchemann's work is a reported link between different uses of letters in 'generalised arithmetic' and Piaget's levels of intellectual development.* (page 50)

Hewitt (2003) considered students' reading of formal algebraic notation and he observed that many errors made by students could be accounted for by the strict left-to-right reading of formally written arithmetic statements. He also considered how students read word statements, acknowledging that *expressing non left-to-right order in written words can be problematic as well since words do not possess a set of notational conventions, such as brackets* (page 34)

In the National Strategies Secondary Mathematics Exemplification (DCSF, 2008) the learning objective relating to this states that pupils should be taught to: ***Use the order of***

operations, including brackets. (page 86) One example of a learning outcome is that a year 7 student should be able to perform the calculation

$$\frac{5^2 - 7}{2^2 - 1}$$

either mentally or using jottings. This objective is also linked to calculator methods, with the expectation that a pupil should be taught to **carry out more complex calculations using the facilities on a calculator** (page 108) and with the order of algebraic operations, where pupils are expected to understand **that algebraic operations follow the same conventions and order as arithmetic operations.** (page 114) The exemplification makes it clear that pupils are expected to be able to use a scientific calculator efficiently when evaluating more complex mixed operations.

The methods for teaching this topic can vary, but one common theme in some countries is to use a mnemonic to aid the memorisation of the order of operations. In the UK this is commonly **BIDMAS** or **BODMAS**:

Brackets Index Division Multiplication Addition Subtraction

Or

Brackets Of (Order) Division Multiplication Addition Subtraction

In the USA the mnemonic **PEMDAS** is commonly used:

Parentheses Exponents Multiplication Division Addition Subtraction

Clearly this may have its uses in remembering the “rule” once the concept has been understood, but it is the clear understanding of the underlying principle and conventions that enable it to be put into practice. This includes the understanding of index notation and the recognition of a fraction for division.

Thus it is far from merely being a case of learning a mnemonic; a sound understanding of mathematical notation and structure is required in order to carry out a calculation of the type given in the National Strategies Mathematics Exemplification. It is this deep understanding that lays the foundations for an understanding of algebraic structure.

It is interesting to note that the use of mnemonics is not referred to at all in the National Strategies documentation, and yet many text books and other resources used in the UK and in the USA encourage it.

In contrast, from conversations with Japanese teachers it would seem that mnemonics are never used in Japan. Indeed in the Japan National Mathematics Program (2000) the order of operations is not specifically referred to at all. The teaching methods are very didactic with a large emphasis on whole-class teaching and repetition of questions, focusing on algebraic structure.

The study: Context and Methods

The primary aim of this study is to examine the ways that pupils perform calculations which require the correct use of the order of operations and to study the misconceptions that may arise. One tool that will be utilised in order to carry out the research will be a piece of software that was developed as a research tool by Texas Instruments in conjunction with the University of Plymouth. This software is called the Key Recorder and can be loaded onto the more recent models of the TI graphics calculator. It has been used as a data collection tool in a small number of research projects (For example: Graham, Headlam, Honey, Sharp and Smith, (2003), Berry, Graham and Smith (2003), Smith (2003) Berry, Graham and Smith (2005), Berry, Graham and Smith (2006), Sheryn (2005), Sheryn (2006a), Sheryn (2006b) Graham, Headlam., Sharp and Watson (2007))

This study involves classes of children who have been taught about the Order of Operations and who would therefore be expected to be able to perform calculations based

upon these principles. These children are in the age range 12 – 14 (Years 8 and 9 in UK schools). The children complete worksheets of questions involving a variety of calculations, some with and some without a calculator. For the calculator-based questions the children are provided with a TI-84 graphics calculator which has the Key Record software running.

When the children’s work has been initially analysed, some of the children are then interviewed in order to follow up and pursue questions that arise.

Initial Findings

A pilot study was carried out in the UK and in Japan. In each country one class of students was involved. In the UK this was a class of 20 middle ability students in year 8 (age 13). The Japanese class consisted of 33 mixed ability students aged 14.

Both classes had been taught the principles of the order of operations as part of their scheme of work, and had also been taught simple algebraic conventions, including substitution of letters for numbers in algebraic expressions.

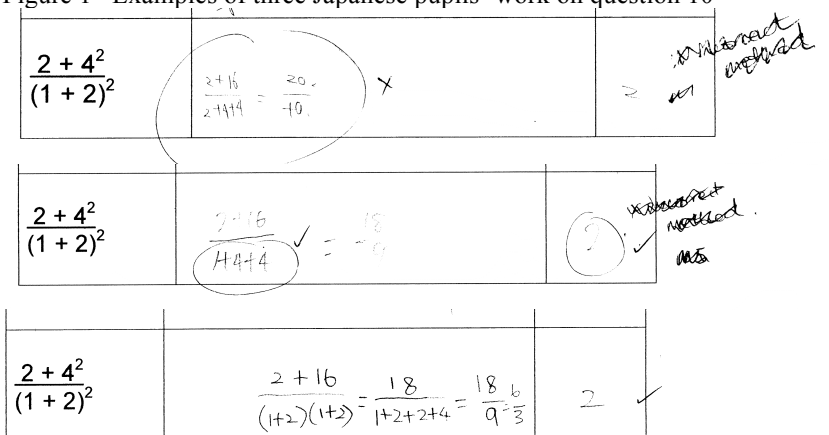
In the pilot study the graphics calculators were not used; the children were given one worksheet to complete without using a calculator. As a result of this study the worksheets were adapted and a second worksheet produced. The second worksheet contained questions which were identical in structure to those in the first worksheet but involving decimal numbers which would encourage the use of a calculator. The children would be given a graphics calculator with the Key Record Software running which they were asked to use when completing the second worksheet. The main study has now been carried out in a further two classes in UK schools, both middle ability year 8 classes. The children completed both worksheets, and afterwards their worksheets was analysed alongside the Key Record data. Some children were then interviewed and the teachers were also interviewed.

From the pilot study it was interesting to investigate the questions that the Japanese children got wrong, and to analyse their ways of working. There was a general tendency to treat all the questions as algebraic, even though they were mainly numerical. The calculation that was answered incorrectly by most Japanese pupils was question 10:

$$\frac{2 + 4^2}{(1 + 2)^2}$$

the calculation $(1 + 2)^2$ was in many cases calculated by expanding the brackets first:

Figure 1 Examples of three Japanese pupils’ work on question 10



and it was observed that the incorrect answers were more likely to be due to careless errors rather than revealing misconceptions. When calculated in this way, the need to remember a rule such as BIDMAS becomes unnecessary, although there is still a need to know that indices are evaluated first in the numerator.

This question was approached differently by the children in the UK. One girl's attempts at questions 9 and 10 of the non-calculator worksheet are shown in figure 2:

Figure 2 One UK pupil's work on questions 9 and 10 on non-calculator worksheet

9	$\frac{8+6}{3+2^2}$	$\frac{14}{25} \times$?	\times
10	$\frac{2+4^2}{(1+2)^2}$	$\frac{36}{9}$	4	\times

In question 10, although she correctly evaluated the denominator, involving brackets, she did not evaluate the index first in the numerator. When interviewed, she was asked what she was thinking about when doing the worksheet, she immediately answered "I was thinking of BIDMAS" but when asked what this stood for she hesitated and then answered *Brackets, individual, divided, multiply, addition and subtraction*. When prompted about what the "I" stood for she did not know, and even when asked about the word "Index" she was not sure what this meant, although when she was shown the number she immediately said "oh - to the power of 2?" which revealed that she understood what a power was, but had not related this to the word "index" and therefore could make no sense of the I in BIDMAS. The same misconception is also revealed in her answer to question 9. Her attempts at the corresponding questions on the calculator paper are shown in figure 3:

Figure 3 The same pupil's work for questions 9 and 10 of the with-calculator worksheet

9	$\frac{8.2 + 6.3}{3.7 + 2.8^2}$	$\frac{14.5}{11.54}$	1.2564991 ✓
10	$\frac{2.4 + 4.8^2}{(1.8 + 2.7)^2}$	$\frac{25.44}{25.25}$	1.256296 ✓

Analysing her keystrokes revealed that she used her calculator efficiently with a good grasp of the need to evaluate each of the numerator and denominator first before dividing. Her incomplete understanding of the BIDMAS rule was overcome by using the calculator efficiently. For question 9 the pupil correctly evaluated the numerator on her calculator, and then evaluated the denominator:

3.7+2.8 ²	11.54
2.8 ²	7.84
Ans+3.7	11.54

Once happy with the denominator she proceeded to finish the calculation:

14.5/11.54	1.256499133
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It would seem that she wanted to check that the answer from the first line was the same as evaluating the power first, then adding, thus indicating that she had an idea of the correct order, even though she had got the equivalent non-calculator question wrong. When she was using her calculator she was able to investigate the effect of calculating the power first and successfully confirm that this was the correct way to carry out the calculation. In

question 10 the pupil now seemed satisfied that the calculator would produce the correct result for both numerator and denominator and worked efficiently to produce the correct answer:

2.4+4.8² 25.44
 (1.8+2.7)² 20.25
 25.44/20.25
 1.256296296

Conclusions

In this first part of the study it would seem that there are substantial differences between Japanese and British childrens' ability to carry out arithmetic calculations and the approaches used. The Japanese pupils relied upon algebraic approaches which they generally employed correctly, but this sometimes caused them to make the calculation unnecessarily complicated and they made algebraic mistakes. The children in the UK relied heavily upon remembering BIDMAS which worked well if they remembered it correctly but broke down if they did not fully understand what all the letters stood for. However the use of a calculator enabled the pupils to experiment and discover the conventions, which reflects the teaching approaches used.

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