Representational strategies of students with difficulties in mathematics: responses to a 'Cartesian product' problem

Carla Finesilver

Institute of Education, University of London

This paper explores aspects of representation in students' responses to a 'Cartesian product' problem presented in story form (Nunes & Bryant, 1996), and is based around multimodal data taken from my current doctoral research project, including scans, photographs and transcriptions. Twelve students were in KS3 mainstream education, one in KS4, and had been identified by their mathematics teachers as the lowest-attaining in their respective year groups, displaying significant difficulties with mathematics. No task-specific materials were provided, but paper, coloured pens and multilink cubes were available for students to use if they wished. Representational strategies included colourful pictorial depictions of the items, physical models both with and without movement into different configurations, and some abstract notations. I am currently developing a framework for analysis, with aspects to be addressed including the level of abstraction found in each of the representations, the ease with which students chose or created strategies, identification of any changes in representational strategy that took place during the task, and the types of support that students required for successful task completion.

Keywords: secondary mathematics; numeracy; Special Educational Needs, visual representation/communication

Introduction

In Children Doing Mathematics (1996), Nunes and Bryant describe a particular type of multiplicative situation which they refer to as a Cartesian product problem. This particular type of problem has been thought (for example, by Brown, 1981, and Nesher, 1988) to be particularly difficult for children. It is most easily described by example, e.g.:

> Mary has three different skirts and four different blouses; how many different outfits can she wear by changing around her skirts and blouses? (Steffe, 1994, in Nunes & Bryant, 1996)

Nunes and Bryant (1992, in N&B 1996) observed 32 8- and 9-year-old children solving a set of multiplication problems, one of which was similar to the example above. The numbers involved were 6 and 4, and some materials were provided: half the children received a complete set of miniature clothing items, while the other half received only a subset. The results indicate that these problems do seem to present difficulties for children: the 8- year-olds' success rate was negligible, while around 55% of 9-year-olds solved the problem with all materials. Nunes and Bryant recommend that "when children are able to solve simple one-to-many correspondence problems, it may be worthwhile to give them more complex problems, where the correspondence is not explicitly stated", for example Cartesian product problems. I followed this suggestion, and incorporated an 'outfits'-style problem in my initial assessment sessions with students.

My 'Holiday Clothes' task was based on the scenario of a child packing for a holiday. I retained the numbers (six and four) except in the case of one student with particularly weak numerical skills, when I reduced them to four and three. I wrote '6 t-shirts' followed by the list of colours, and similarly for trousers; I then gave two verbal examples of combinations (the blue trousers with the blue t-shirt and the blue trousers with the white t-shirt, chosen to reinforce the idea that each item could be used more than once). The main difference in my version of the task was that, in line with the larger project of which this is a part, I did not pre-prepare any problem-specific materials for students to use; instead, I provided coloured felt-tips and multilink cubes, and indicated that students could make use of these in any way they found helpful.

Student responses to the task

One Year 9 student, Wendy, used no external representations in her solution of the problem. However, it should be noted that this does not imply ease of solution; she still took ten minutes to arrive at the correct figure, after several incorrect suggestions. From her verbal responses, it seems that she was making use of internal visual representation; however, discussion of internal representations is outside the scope of this paper. The other twelve students all used writing, drawing, modeling with cubes, hand gesture, or some combination of the above. Examples of students' work are organised in this manner for presentation.

Writing

Figure 3: Jenny Y7

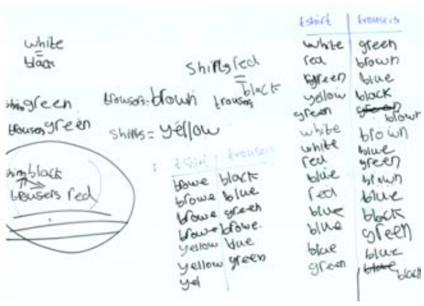

Figure 2: Tasha Y8

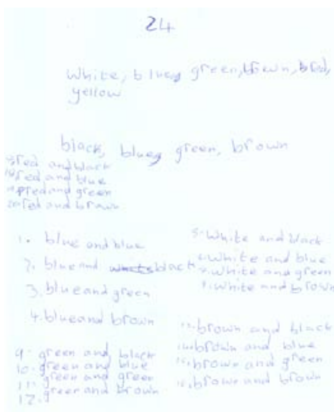

Figure 1: Danny Y8

Jenny began by choosing a colour combination and listing it both ways round [Fig.1]. At first, colours were not worked through systematically, but then clusters appeared, the list becoming more systematic as she continued, to the point where she looked for missing combinations from colour subsets. Jenny has made a key error in reversing the colours for all her t-shirt/trousers pairs, when it is not actually possible to do so; however, this could be taken as evidence of abstraction, the divorcing of the mathematical aspect of a problem from its original 'physical' scenario.

It is surprising that Tasha chose a written strategy [Fig.2] as she generally preferred working with cubes. At several points she requested and received help; for example, when she found the writing too laborious, I suggested a table. She became frustrated on repeatedly asking if it was "finished yet" when it was not. However, when I asked if she had all the different outfits that included a blue t-shirt, she quickly completed the set, then checked and complete the rest in a quite systematic manner.

Danny paused for a minute to consider the problem, then worked without support. His response [Fig.3] differs clearly from the two other written strategies in that it shows an immediate grasp of the problem's Cartesian structure, as demonstrated in the orderly progress of combinations from the start. However, two points are of particular interest. Firstly, there is clearly a distinction to be made between (a) comprehending the structure of the problem and (b) realising that all that needs to be done is to calculate six times four. This is perhaps not obvious.

Figure 6: Oscar Y9

Figure 5: Kieran Y7

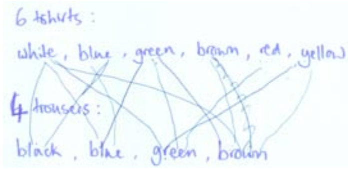

Figure 4: George Y8

Drawing

Oscar's (unfortunately unfinished) piece [Fig.4] represents five minutes spent on the task without pause or comment. Other than beginning with 'matching' pairs, there does not appear to be any pattern in the combinations listed.

Kieran [Fig.5] barely paused between the first four drawings, then made a significant strategic change and reduced the amount of drawing necessary by drawing the four colours of trousers below each t-shirt. His solution shows 18 outfits, and the lesson ended before he could check his work thoroughly.

Although George's mode of representation [Fig.6] is drawing, there is a fundamental difference between his and the previous two (or in fact any of the responses seen so far), in that he does not represent the individual clothing items at all, but draws the relationships between them. As with Danny, he demonstrated a grasp of the problem's Cartesian structure, and was able to represent the problem scenario in a more abstract way; however, again, this did not lead to a numerical calculation. George's difficulties are apparent, with some links missing or repeated, errors which he was unable to see without my support. This suggests perhaps some kind of issue with visual processing (an area for future investigation).

Figure 7: Sidney Y9

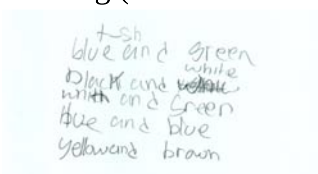

Figure 8: Sidney Y9

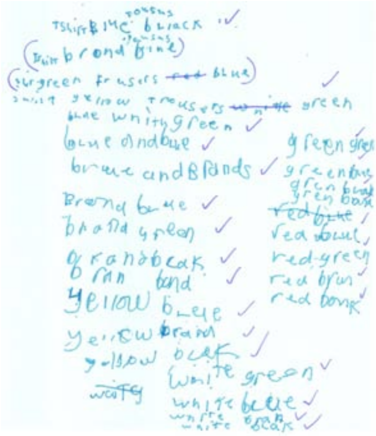

Figure 11: Harvey Y7

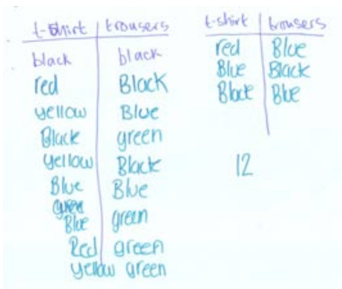

Figure 10: Paula Y10

Figure 9: Paula Y10

Modelling

Sidney had only a short time on the task. He began by listing two pairs in written format, then asked to use cubes. He then made the pairs shown [Fig.7]. Although the visual stimulus of coloured cubes was instrumental in Sidney's progress with the task, a potential issue is highlighted in the fact that with two modes of representation in use (effectively, one for thinking about the problem and one for recording), there is the constant need to translate between them, and in this case, a translation error occurred. Thus multiple representations can be both a help and a hindrance.

Harvey began by listing four pairs of clothing items [Fig.9] before stopping. His first few answers included both valid and non-valid items (e.g. red trousers), which I pointed out. On his request for help, I offered a prompt by taking 10 cubes, in two groups corresponding to the 6 t-shirts and 4 pairs of trousers, then placed a black cube next to cubes of the different t-shirt colours in turn, reinforced verbally. After this, he was able to use this system of picking a t-shirt colour and completing the set of four, with occasional prompts. When he reached the end of his list, he immediately said "Finished!" with confidence, which indicates to me that he was aware that all the possible combinations were now exhausted.

Paula tended to find multiplicative structures extremely hard, so I set her the Cartesian product problem with smaller numbers, and was prepared to give her a higher level of support, which was indeed required. I drew a table for her [Fig.10], and demonstrated how she might write combinations. As she suggested combinations, I made them from pairs of cubes. Paula did not recognise that there would be any pattern governing the list of possible combinations, so was unable to systematically check for 'missing' combinations. My response was to place all the cube pairs in a visuo-spatial sequence, arranged by colour [Fig.11]. I left gaps in the appropriate places, explained that some combinations were still to find, and first asked "What should go here?", then made my questioning more explicit, verbally and gesturally, in reference to the pattern of colour pairs, i.e. "We have the blue t-shirt with the blue trousers, the blue t-shirt with the green trousers; what else must the blue t-shirt go with?" At this point she was able to identify the remaining combinations.

Other responses

Ellis was the fastest student to complete the task, taking around two minutes. He saw the Cartesian structure of the problem quickly, but like Danny and George, did not recognise it as six times four. His representation was of the relationships between the two sets of items, but rather than draw in the links as George did [Fig.6], he simply used a finger to trace them out, counting aloud as he did so.

In some of the previous examples, students made strategic changes during their completion of the task, to work more systematically or efficiently.

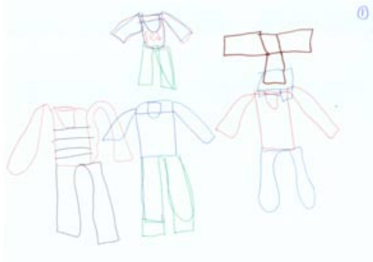

Figure 13: Leo Y7

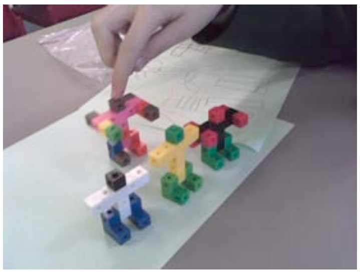

Figure 12: Leo Y7

However, two students (in a paired session) made distinct changes in representational mode without suggestion from me, discarding their original strategies.

Leo began by choosing to draw [Fig.12]. However, he ran into difficulties because his favourite pen was a four-colour ballpoint and he would not use my

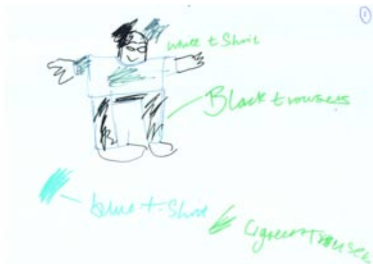

Figure 14: Vinny Y7 (30 mins)

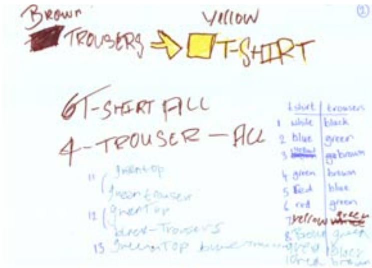

Figure 15: Vinny Y7

coloured pens. (It is noted that Leo is listed on his school's SEN register as having Asperger Syndrome.) He chose to switch to cubes [Fig.13]. However, during the making of his elaborate models, he was taken by the idea that they looked like 'Transformers', and started to play with them, after which it was not possible for me to draw him back to the mathematical task.

Vinny began with an elaborate drawing, both coloured-in and labeled [Fig.14]. He reduced it to 'swatches' of colour, but also writing the names of the items in the appropriate colour [Fig.14-15]. He then decided just to write down the outfits, but requested my help. I drew the table [Fig.15], including his outfits so far, after which he was willing to take over the writing. It is possible to see the emergence of system, with his listing together all the 'green top' pairs.

Aspects for analysis

Many of the representations in my research are co-created by students and myself, and so it is highly relevant the parts of each representation which may be attributed to student and to teacher/researcher, in addition to which there are any verbal prompts offered. As support was given to some students in their completion of the task, the nature of this must also be considered: in this case whether it related to understanding the structure of the task, to the recording of combinations or to their enumeration.

Two aspects of representational strategy have already been mentioned: *Mode* and *Media*. On this task the modes of representation used were writing, drawing, modelling, gesture, and the media employed pen & paper, and cubes. When considering the gestural mode, it may be useful to think of hands and anything currently held in them as media for gestures. In addition, many other aspects are emerging from the data collected, and I am currently developing a full analytical framework. Probable dimensions of analysis include the following:

Motion. Modelled representations in particular may be static (e.g. Paula, Fig.11) or include motion (e.g. Harvey, no photograph). In this task, both have advantages: the static model gives an immediate image of all combinations, with the colours enabling visual pattern-spotting, whereas using one cube per clothing item was closer to the actual task situation, but required notating in some way (assuming the multiplicative structure had not been recognised).

Completeness. On this task, *completeness* refers to whether a complete solution set of combinations was required to be present and visible for the total number to be found. This can refer to whether all combinations are visible simultaneously (as in previous examples), or to whether a student working systematically through the combinations needed to write out the complete list, or 'jumped' to the answer at a certain point and left the representation 'incomplete'. Note that *completeness* is not a positive attribute or the opposite, 'incompleteness', a negative one (e.g. Danny, Fig.3, who did not need to write a complete list to work out the answer).

Resemblance. This refers to how much the working representation visually resembles the situation specified in the original task presentation. For example, it may be pictorial (e.g. Fig. 14, where Vinny draws a man wearing the clothes), iconic (e.g. Fig.5, where Kieran draws simplified t-shirts and trousers), symbolic (e.g. Fig. 7, where Sidney uses a blue cube to stand for 'blue clothing item'), or schematic (e.g. Fig. 6, where George draws the relationships between the items rather than the items themselves).

Consistency. A representational form may be used consistently from start to finish, or the student may make changes during the representational process (e.g. Kieran, Fig. 5, changing from drawing individual t-shirt/trousers pairs to a more efficient grouped format). Thus, like *completeness*, *consistency* is not necessarily a positive attribute.

Concluding remarks

Analysis is currently at an early stage, but already it is clear that these students, selected for their low prior attainment, are capable of great variety and inventiveness of approach to tasks, and, with encouragement and the right representational support, can engage with more sophisticated arithmetical structures than might be expected.

References

Brown, M. 1981. Number Operations. In *Children's Understanding of Mathematics 11- 16*, ed K. Hart, 23 - 47 Windsor: NFER-Nelson.

Nunes, T. and P. Bryant. 1996. *Children Doing Mathematics*. Oxford: Blackwell.