Generalisation and perceptual agility: How did teachers fare in a quadratic generalising problem?

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This study examines the perceptual agility and strategy use of 27 prospective secondary school teachers in Singapore when solving a quadratic generalising problem. The data showed that the teachers were very capable of employing a variety of strategies to visualise the same pattern in different ways, resulting in not only a diverse range of equivalent rules but also some creative visual representations of the pattern structure.

Keywords: Pattern generalisation, Perceptual agility, Prospective teachers

Contemporary context

One of the essential 21st century skills calls for individuals to think creatively and to solve problems using both conventional as well as innovative strategies (Learning and Skills Development Agency 2003). Particularly relevant to mathematics, this skill finds an academic home in the topic of pattern generalisation, which is about seeing the general in the particular. The idea of having to develop flexibility in making generalisation is not a new issue and for some time now, its importance has been emphasised by many researchers (Mason, Graham, and Johnston-Wilder 2005; Lee 1996). Lee (1996) calls the ability to see a pattern in multiple ways as perceptual agility. With even greater emphasis in today's society, it seems reasonable to think that perceptual agility is not a luxury but a must-have for all students, rather than just for the more-able students. However, to nurture students and help them develop this ability, teachers must become versatile in engaging in different ways of seeing a pattern so as to be able to provide proper guidance and model how to do it. But are teachers perceptually agile themselves? This paper aims to investigate this issue through the asking of the following two questions: (1) Are prospective teachers capable of deriving multiple expressions for the same generalising problem? If so, to what extent? (2) What strategies did the prospective teachers employ to derive the expressions for the generalising problem? The article begins with a discussion of the methods, the classification scheme used for data analysis as well as results, and follows with a discussion of research findings.

Methods

A 90-minute written test was administered to 27 prospective secondary school teachers at the National Institute of Education in Singapore. One of the test items is the Expanding Cross task presented in Figure 1. This generalising problem was chosen because (1) it involves a quadratic rule in one variable rather than the usual linear function in one variable, and (2) it is less structured, thus allowing a greater scope for exploring the pattern structure. The teachers were asked to solve the problem individually first, and then in groups to generate the rule in as many different ways as possible. In total, there were 13 groups of teachers: 12 pairs and a trio.

Calculator use was permitted for numerical computations since it did not alter the original intent of the problem.

27 test scripts were collected and then marked to check that each solution led to a correct expression of the rule. After marking, the number of correct solutions produced by each group was counted, and a detailed solution-by-solution analysis was then carried out to examine the teachers' approach to establishing the explicit rule in each solution. Details of the classification scheme used in the analysis are elaborated in the next section. With this classification scheme, all solutions were analysed thoroughly again and coded accordingly. A few days later, they were coded separately again to ensure consistency in the coding process. Any discrepancies in coding were carefully examined and negotiated until they were finally assigned to the category that best matched them.

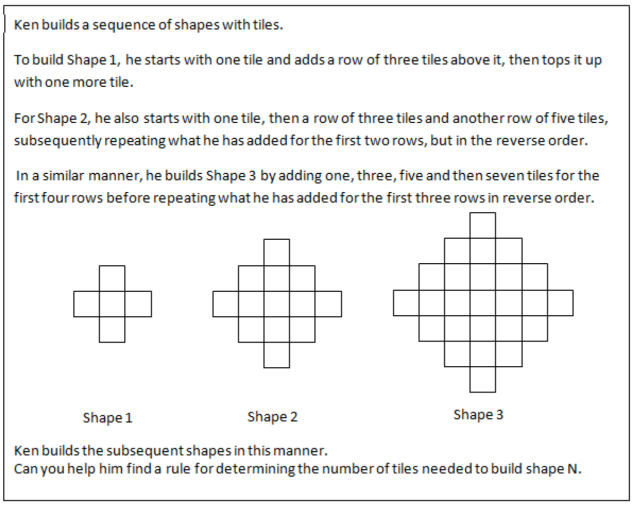

Figure 1. Expanding Cross task

Classification scheme

In the classification scheme that Becker and Rivera (2006) had developed for analysing strategy use in pattern generalisation, there are three types of strategy use: (1) *numerical*, which uses only cues established from any pattern that is listed as a sequence of numbers or tabulated in a t-table to derive the functional rule, (2) *figural*, which only applies in generalising problems that depict the pattern using diagrams, and rules are formulated using only cues drawn directly from the structure of the figures given in the pattern, and (3) *pragmatic*, which is engaged when a combination of both the numerical and figural approaches are used in the expression of generality.

Rivera and Becker (2008) further distinguished the solutions that employed the figural approach into two new categories: (1) *constructive generalisation*, which occurs when the figure given in a generalising problem is perceived as a constellation of components combined in an additive and non-overlapping way, from which the generality can be directly expressed as a sum of the various components, and (2) d*econstructive generalisation*, which happens when the diagram is visualised as being made up of components that overlap, and the generality is expressed by separately counting each component of the diagram and then subtracting any parts that overlap. Examples of these various types of strategy use will be illustrated in the next section.

Results

Table 1 presents the number of correct solutions produced by the 13 groups of teachers. As shown in the table, the teachers produced two to five solutions per group, yielding 43 correct expressions altogether for the Expanding Cross task. Seven groups came up with three ways of deriving the general rule while two groups

Table 1
Number of different strategies

Number of correct solutions produced	2	3	4	5
Number of groups	2	7	2	2

each with two, four and five ways, making 85% of them producing three or more solutions.

In total, 13 different equivalent expressions of the general rule were observed among the solutions, with some occurring more frequently than others. The top four most common expressions produced by the teachers, the frequencies and total percentages of occurrences for each expression are presented in Table 2. The remaining nine expressions were seen once or twice only.

Table 2
Top four most common expressions

	Different expressions of the rule	Number of occurrences
1.	$2(n^2) + 2n + 1$	16 (38%)
2.	$n^2 + (n+1)^2$	7 (17%)
3.	$(2n+1)^2 - 4\left[\frac{n(n+1)}{2}\right]$	5 (12%)
4.	$2n(n+1) + 1$	4 (9%)

The strategies used by the teachers to express the general rule will now follow. Figure 2 shows a *numerical* solution, in which the teachers counted the number of tiles in each array, recorded the numbers as a sequence of terms, and then worked out the first and second differences between consecutive terms. Spotting the constant second difference, they recognised that the pattern involved a quadratic rule. Subsequently, the rule was found algebraically by first letting the rule be $an^2 + bn + c$, followed by comparing and equating algebraic expressions with the numerical cues to solve for *a*, *b* and *c*.

Unlike the numerical approach which followed a certain algorithm, the *figural* solution shown in Figure 3 relied totally on visual cues established from the diagrams to derive the rule. By shifting up one or two constituent components of each original cross to form two adjoining squares, it became very clear that the structure of the original pattern was equivalent to a sum of two squares, hence the rule $(n+1)^2 + n^2$.

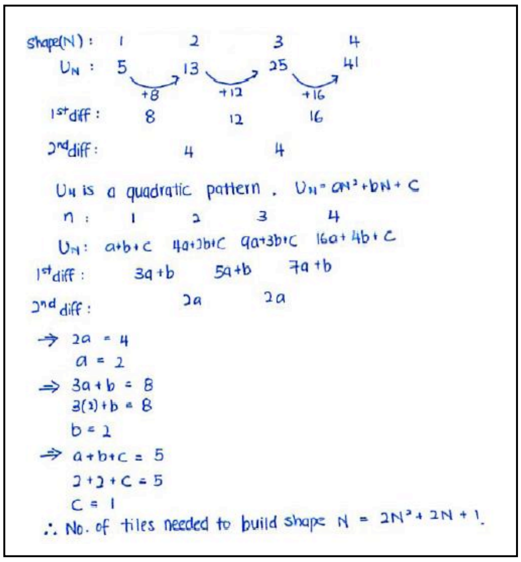

Figure 2. Numerical Approach

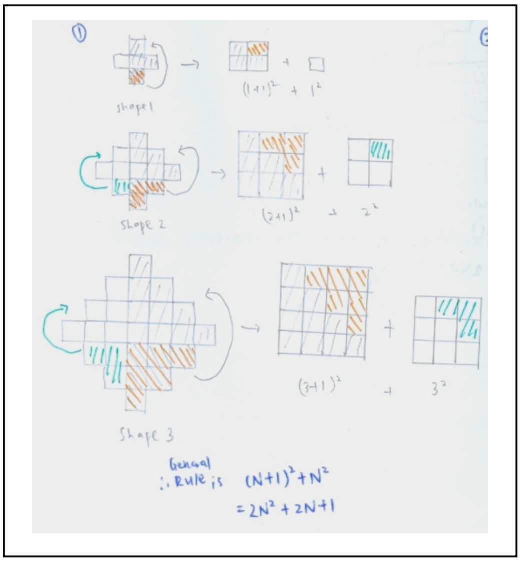

Figure 3. Figural Approach

In Figure 4, the *pragmatic* solution started with a figural approach but later transitioned to a numerical approach. Each cross was seen as comprising three parts: a median row and two identical pyramid-like blocks. By observing that "*2 tiles are added to the structure each time and it started off with 1 tile for the zero term*", the teachers established the general rule for the median row: $2n + 1$. This statement holds crucial evidence of their awareness of an invariant in this pattern – that single tile in the so-called zeroth term to which two tiles are added each time. As for the pattern in the pyramid-like block, the sequence {1, 4, 9, …} was probably obtained through a systematic counting of the number of tiles. Without further relying on any visual cues, the teachers converted 1 to 1^2, 4 to 2^2 and 9 to 3^2, upon seeing a link between the shape number and the number of tiles. This then helps them to obtain the expression n^2 as a generalisation of the number of tiles in each block. By joining the three parts together to form the original cross, the rule is thus given by $2(n^2)+2n+1$.

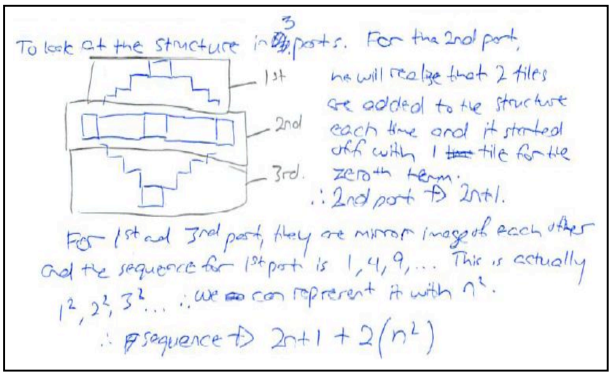

Figure 4. Pragmatic approach

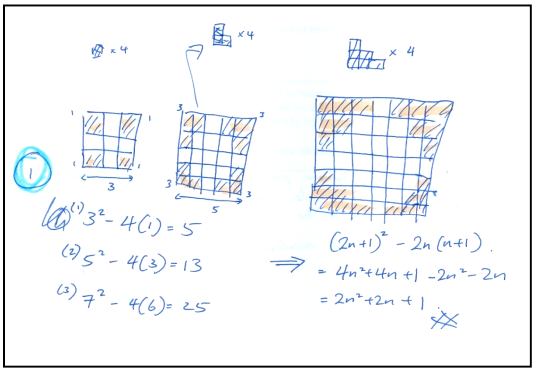

Figure 5. Non-additive constructive

Strategies involving the use of figural cues can be further classified into constructive and deconstructive generalisations. For instance, Figure 4 shows that by viewing the cross as a composite figure made up of non-overlapping components, the rule can be expressed as a sum of the various sub-components. Such a strategy was originally known as a constructive generalisation, but we decided to rename it as an *additive constructive* generalisation in order to introduce a new sub-category, which we called *non-additive constructive* generalisation. The latter involves perceiving the given figure as part of a larger composite figure and then producing the rule by subtracting the sub-components from this composite figure. Figure 5 illustrates an example where the cross is obtained by removing four tiered sub-components from the corners of the large square. With $(2n+1)^2$ tiles in the large square and $\frac{1}{2}n(n+1)$ tiles in each tiered corner in Shape *n*, the number of tiles in the cross is thus given by the general rule $(2n+1)^2-4\left[\frac{1}{2}n(n+1)\right]$, or $(2n+1)^2-2n(n+1)$ when simplified.

Figure 6 provides an example of deconstructive generalisation. Each cross is viewed as being formed by two identical pyramid-like blocks, where one is inverted, overlapping at the median row. So the number of tiles is found by first adding the two identical blocks, followed by removing an overlapped row, yielding $2(n+1)^2-(2n+1)$. This strategy was, however, not observed in this study.

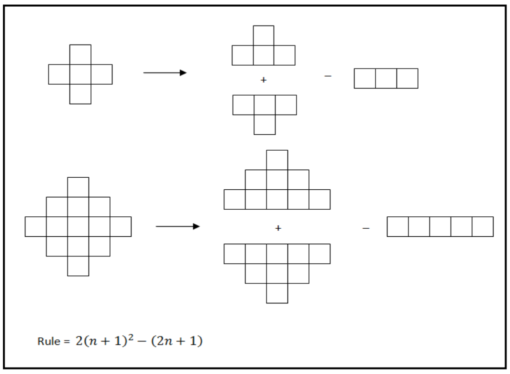

Figure 6. Deconstructive generalisation

Apart from these two kinds of strategy, we discovered that some teachers rearranged certain components of the original diagram to produce a new figure from which they not only see the pattern structure but also subsequently construct a general rule based on this newly reconfigured figure (see Figure 3). Since the classification scheme developed by Rivera and Becker (2008) does not include this strategy, we propose adding a new category to account for such an approach that involves reconfiguration of the original figure. We call it *reconstructive generalisation*.

Reflections and conclusion

Are prospective teachers capable of deriving multiple expressions for the same generalising problem? If so, to what extent?

With a mode of three solutions per group and 85% of the groups giving three or more solutions, the findings seem to suggest considerable evidence supporting the view that the teachers were not only capable of seeing the same pattern in multiple ways when solving the *Expanding Cross* task, but had also done it very well. Despite displaying perceptual agility, success did not come easily to them. They did struggle a fair bit initially before managing to solve it, as the following remarks of two teachers show.

> Teacher 1: We could not find a way to divide the figure into easy-to-manage shapes until after a long time.
>
> Teacher 2: I could not visualise the relationship between the shape number and the number of tiles in the shape. It took me a while to realise that…if I split the shape into three different parts…the question became a lot easier after I did it this way.

It is unclear why the mode is three but we suspect the judgement of how many solutions to give depends on teachers' intuition. When the number of solutions to be given is not made explicit in a task like this asking for multiple solutions, teachers probably gauged that two was too few and five was too many, so three or four was just right. Many might have settled with three eventually, thinking it was good enough, or due to a lack of time.

What strategies did the prospective teachers employ to derive the expressions for the generalising task?

Evidence from the study indicates that teachers rarely used the *numerical* approach, preferring to use either the *figural* or *pragmatic* approach. The relatively low frequency of *numerical* solutions is not at all surprising given that a quadratic generalising task is more complex to deal with than a linear task. Finding a quadratic rule by the typical recursive method is not a straightforward task, like finding a linear rule. Additionally, the use of table will not really help much because the connection between the shape numbers (i.e., the independent variable) and the number of tiles in the corresponding shapes (i.e., the dependent variable) is not easily established. For the few who succeeded in employing the *numerical* approach, it is uncertain whether they are able to provide a valid geometrical justification of the rule even though they were competent in executing the algorithm mechanically to obtain the correct rule. This doubt emerges especially when the visual representation of the rule is entirely disconnected from its symbolic form – a concern which Noss, Healy and Hoyles (1997) had previously raised as well.

Data from the study show that additive constructive generalisation was most commonly used (about 50%), followed by reconstructive generalisation (about 30%), and then non-additive generalisation (about 10%) whereas deconstructive generalisation was not observed. These findings lead to the question: what has contributed a high incidence of additive constructive generalisation and none of deconstructive generalisation? We speculate that the reason why additive constructive generalisations proceed readily lies with a very fundamental mathematical concept – the part-whole relationship, which is first introduced when learning to count whole numbers in primary schools. Thus in this study, perhaps owing to the teachers' familiarity with this concept, part-whole reasoning is naturally triggered when thinking about how the cross in the task (i.e., the whole) is formed. This explanation seems to harmonise with the reason suggested by Rivera and Becker (2008):

> perhaps it is the case that [the children's] constructive generalisations…map easily onto their current understanding of what numbers are and how such entities are used, represented, and manipulated (p.73)

On the other hand, the low to moderate occurrences of reconstructive and non-additive constructive generalisations might be attributable to the complex acts of visual perception. Reconstructive generalisation appears more complicated to visualise than additive constructive generalisation as it requires higher sophistication of spatial visualisation whereas for non-additive constructive generalisation, it can be quite a challenging experience for participants to imagine the missing sub-components in the larger composite figure. As for deconstructive generalisation, its absence here is consistent with the findings from an earlier study conducted by Rivera & Becker (2008), so it is not surprising. The teachers' unawareness of such a strategy may have limited the occurrence of deconstructive generalisations.

In conclusion, findings drawn from this study had shown that the prospective secondary school teachers had demonstrated a remarkable capability of engaging different ways of seeing the same pattern in a quadratic generalising task, and of producing a diverse range of equivalent rules for the same pattern. We discovered that most rules were expressed using cues established directly from the pattern structure, involving several cases of seeing the cross as being made up of different parts combined in an additive and non-overlapping way. We encourage future study to investigate whether prospective teachers continue to manifest perceptual agility in other generalising tasks. If there is a range of equivalent rules, are teachers able to recognise which rule is more mathematically useful in describing the pattern structure, an important issue raised by Lee (1996)? In addition, further examination of the teachers' ways of visualising the figures could yield valuable insights into their interpretations of the pattern structure. The findings will then help to shed light on why certain strategy use occurs more frequently than others.

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