The Validation of a Semantic Model for the Interpretation of Mathematics in an Applied Mathematics Problem

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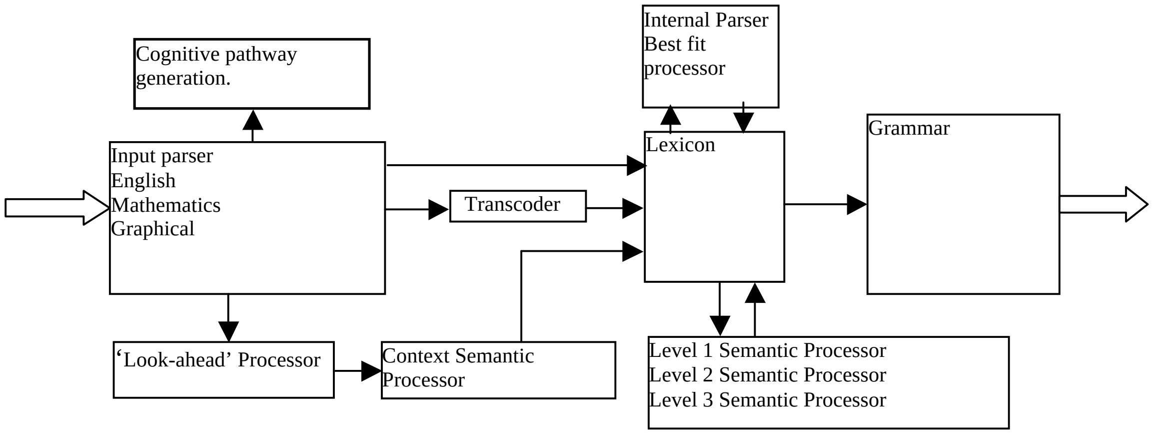
The semantic model proposed by Peters (2008) was developed whilst working with learners of mathematics solving algebraic problems. In order to investigate in more detail the role of the parsing process and its relationship to the lexicon, a different set of questions were devised based on Laurillard's (2002) work with undergraduate students. These same questions were also given to a set of mathematics tutors so that a comparison could be made between the two groups and to see if their behaviour could be explained using the semantic model. The analysis of these sets of data indeed show the importance of the parsing process and as predicted by the model, a competent mathematician employs a top-down parsing strategy.

Keywords**: Parsing, Lexicon, Cognitive Pathway.**

Introduction

The algebraic problems from the Chelsea Diagnostic Test (CDT) used in research that led to the development of a semantic model of the processing of mathematics (Peters, 2008) were all situated in a mathematical context and were abstract. A problem Laurillard (2002) was presented to undergraduate students to validate the semantic model and investigate the role of the parsing process and its relationship with the lexicon; it could be represented in a form of words, as a diagram and be interpreted using experiences of 'everyday' life. There were two scenarios: (1) a box resting on a table and, (2) a box in mid-air. Students were asked to explain the scenarios using Newton's Third Law of Motion. For the purpose of this investigation this problem was broken down into four separate questions; this required the learner to parse the orthographic form, the graphical form, access their lexicons and retrieve the appropriate entry.

The Semantic Model



The figure shows the semantic model developed by Peters (2008). This model was based on ones developed for arithmetic (e.g. McCloskey et al 1985), natural language processing (Chomsky 1981) and work by the author on how learners parse mathematical structures. It models the processes a learner of mathematics uses to interpret and make sense of mathematical problems. The process starts at the left where the learner reads the problem. The input parser is used in the initial stage to analyse the syntax of the problem and start the process of deriving the semantic content. If the problem is in a mathematical form i.e mathematical equations or expressions, the lexicon can be accessed directly but, if it uses a graph or natural language to set out the problem, a process of transcoding is used to alter the input into a form suitable to access the lexicon. The output from the lexicon works with the grammar to select the appropriate rules to resolve the problem. The initial parse of the problem also highlights the need to scan ahead and determine the context of the sign and ensure an appropriate semantic interpretation is attached. The cognitive pathway generation is the process where, if the problem is a familiar one, the relationship between the lexicon and grammar is known and a reduction in cognitive load can be achieved. If the form of the problem has not been encountered before a cognitive pathway has to be initiated. This pathway takes the form of developing the links between the cognitive modules. The semantic processing is depicted with three levels which arbitrary in the sense of a problem that was difficult to solve initially by the learner requires careful semantic interpretation whereas once the learner is confident with such problems the amount of semantic processing is reduced.

Research Methodology

The learners and the tutors were presented with the questions in a written format and in a specific order: vague to more detailed. Once a question had been presented and the answers given, the scripts were collected and the participants interviewed. This method was adopted so the more detailed formulation of later questions would not influence their responses to previous answers when interviewed. The interviews were conducted in pairs so that the discussion between both participants could be observed and analysed. The interviews were recorded and later transcribed for future analysis.

Validation of the Semantic Model

Question A: Explain how Newton's Laws of Motion apply to the two situations shown in Figure 2.

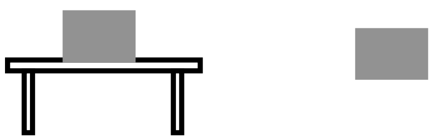


Figure 2, (a) A box resting on a table. (b) A box in mid-air.

This question was deliberately vague in respect to which law(s) of motion could be applied.

The written and verbal answers of the learners indicated that the initial parse of the question created problems. Learners who were unable to recall the laws indicated a failure between the parser and the lexicon. Their parsing of the question failed to identify the appropriate lexical entry, since the information gathered in the parsing process did not provide them with the necessary key information to facilitate the link between the parser and the lexicon. One learner's explanation, for which he did not state any assumptions, was that the box in mid-air must have been filled with a gas in order for it to float. Other learners stated assumptions such as; ignoring air resistance, the box was falling, model the box as a particle. In their written explanations all learners stated the box would fall due to gravity. None of them mentioned the force of the box on the Earth, but the notion of equal and opposite forces was mentioned in conjunction with the box resting on the table.

When asked what they interpreted to be the key words, the answers included '*explain', 'Newton's laws'.* One learner in the interview stated that once she had read the question her main worry was trying to remember Newton's laws of motion. She said that she thought she should have interpreted it in an 'everyday' context but did not do this because *"I was totally fixated on the maths, because that's what we've been doing."*

When asked if they could recall or if they had a vague idea of what they wanted, one of the learners responded with

> "yes I had a vague idea of what I wanted and I was cross with myself that I didn't know the topic well enough to be specific, because we've done it relatively recently."

One learner decided the third law was appropriate and gave a good explanation of the relationship between the forces in the box resting on the table. When it came to the box in mid-air, he could not identify a similar relationship. He knew gravity was 'pulling' the box towards the Earth and for the law to apply, he needed to be able to identify another force which he was unable to do. In the end he decided that air resistance provided the other force even though in his assumptions he opted to ignore air resistance.

Question B: Explain how Newton's Third Law of Motion applies to the two situations shown in figure 2. This question gave the learners more information; it told them specifically to use the third law which should have acted as a trigger for their lexicons. One of the learners underlined 'Third Law of Motion' in the question and in her explanation wrote "*Box on table exerting same force down as table exerts up ?3rd law."* In the interview, when asked her immediate thoughts, responded with *"What is the third law of motion. Is it the one I described in the last piece of paper and I don't know."* This response indicated they recognised the importance of the words (the key), '3rd law of motion', but a strong cognitive link had not be formed between the parsed words and the lexicon. Similarly one of the learners in the other pair responded with *"What's Newton's third law and that scared me, because I just didn't really know."* He went on to say:

> "...although it's the same question and I've already been influenced by question sheet A...but because it said third law, it's just like well I don't even know if I know the third law...and then I was thinking well, if the table was removed would the box just stay there, is it the same box, in which case the table is irrelevant..."

This learner also stated that once he read 'the 3rd law', he started panicking. When asked to explain what effect this panicking had on him, he stated:

> "...that's why I went into like freefall, that's when I started thinking about the table and things like that, cause it takes my mind off my stumbling block and I'm just trying to see if I can get around, there's another pathway to the answer that I want and I couldn't find it."

Question C: Explain how Newton's Third Law of Motion which states: 'every force has an equal and opposite reaction', applies to the two situations shown in figure 1.

This question gave learners a prepared statement of the third law in its canonical form.

The learners' written answers to this question included words such as 'force, reaction, equilibrium, at rest, forces cancel' whereas in question A and B they used 'force, equal, opposite, gravity'. This change in vocabulary indicated that they did have lexical entries for the more mathematical terms used in question C. The problem with the wording of the third law is that it omits any reference to the fact that the definition of the law requires that there exist two bodies exerting forces of equal magnitude on each other. As Laurillard (2002) pointed out the phrase 'equal and opposite' implies that the forces cancel out to give equilibrium which gives rise to the learners' misconception about the third law.

One learner gave a succinct answer: "Gravity pulls box down. Table pushes box up. Forces cancel and box stays still." Another answer was: "Normal reaction of the table balances the weight of the box." Their answers to scenario (b), the box in mid air, remained in essence the same as their previous answers except that the word 'force' was now used. For example, one learner stated:

> "If box is stationary must be some sort of upwards force to counter balance box force downwards otherwise this will not last long in mid-air."

Another responded with: "Gravity pulls box down. Force inside box pushing up – assume lighter than air gas. Forces cancel and box stays still.

One learner attempted to answer the two possible situations: the box moving towards the Earth and the box 'floating' in mid-air. His answers were:

> "Gravity is the only force acting on the box' and 'if the box is in mid-air (floating) then lift should equal mg."

Another learner in an attempt to resolve the point of ambiguity resulting from her interpretation of diagram (b) suggested:

> "If we assume the box is in mid-air is in a vacuum it will have no forces acting on it so will stay where it is."

Question D: Explain how Newton's Third Law of Motion (When one object exerts a force on a second object, the second object exerts an equal and opposite force on the first), applies to the two situations shown in figure 1. The problem with this question for the learners was the word 'object'. In part (a) they could explicitly see the objects involved whereas in part (b) only one is shown. Their answers, both written and oral, supported the view that there was only one object present in part (b) i.e. the box.

One learner in the interview stated in response to being asked if the question was different:

> "Yes very different, because you put the word objects in. So you said one object is exerting a force on a second object and going back to my poor box in mid-air it has no other objects..."

The other learner in this pair correctly identified the Earth as the other object but this just created a point of ambiguity.

> "...I got confused and couldn't decide whether it was relating to this law of motion or another law of motion which would account for situations...as you were saying (referring to other interviewee) the Earth is a factor that should be taken into consideration...I'm trying to trawl through all the laws and try and work out that there's other things that are equal and opposite but they don't happen to have objects in them."

In their written explanations one of them wrote: 'Hadn't considered objects so important but I suppose other motion laws account for other situations'. The other wrote '...still not happy with the box in mid-air, there's no second object...'. The notion that the Earth can be considered as an object seemed difficult for them to grasp. Two of the learners confirmed this when asked what they considered to be the problematic words:

> "B: The key word is object, different word...makes you think in a different way.
>
> A: I think you immediately think of an object as something...(B interrupts).
>
> B: Something tangible, three dimensional you can see and feel...it was almost superfluous because when we looked at question C we almost assumed that...we did not really need to be told they were objects as well...and we're thinking of the Earth and gravity and somehow you don't think of them as objects in the same way (referring to question D)."

One of the learners had difficulty in the overall structure of the formulation of the third law. When asked about the question in general, he replied:

> "Too wordy. I struggled to read it, too many objects within a sentence and Newton's third law the way it was explained in the question was just too wordy and it took me too long to try and work out what the third law was, even though I already knew from the previous question."

Analysis of Tutor Answers

Initially the tutors could not see the point of the question since it was obvious to them how the scenario should be resolved. This indicated that their lexical entry for the situation was very well developed. Although they parsed the question, the diagram was used as the main source of obtaining the necessary information.

In response to question B:

> "…But I can't recall which one we call the first and which one we call the second…I don't think, to be honest, I don't think it really makes any difference when you come to analyse a problem as long as you know those two concepts."

The tutors did not question the box in mid-air; they immediately assumed the diagram was a 'snap-shot' and consequently the box was accelerating towards the Earth; they did not debate how the box could be suspended in mid-air. This was clarified in response to question C when asked what assumptions they made:

> "Well, if it's in mid-air, it's gotta move. If it's in a gravitational field of any kind."

It seems as a part of their parsing process, assumptions were automatically generated and any unrealistic situation was immediately discounted. They assumed implicitly that the scenario was Earth based and a gravitational force existed.

Summary

The responses from the learners who participated in this exercise highlight the difficulties that arise from underdeveloped lexicons and the parsing process. In terms of my semantic model it seems that when they parsed question A the cognitive pathways were initialised. Question A was not specific and did not mention the third law and learners used their existing conceptions. As the questions became more specific, these conceptions were challenged resulting in ambiguity which needed to be resolved. When asked, if they were in an exam, which question they would be most comfortable with, they replied question C. The formulation of the third law in this question was probably one they were familiar with and therefore they considered they understood it. The questions also highlight the difficulty learners have relating everyday experience with the mathematical interpretation of phenomena. It was apparent from this study that once the learners had reinforced the cognitive pathways initially set on parsing question A (reinforced by question B since it did not challenge their conception of the third law) they had difficulty in resolving the scenario in a mathematical context. It seems that they attempted to make the mathematics fit their conceptions rather than reparse and reset their cognitive pathways.

This study also highlighted the importance of wording questions in terms that are understood by the learner. For example, in this study it was assumed the learners would know what was meant by ‘object’ in question D and that they knew that gravity was a force. This particular problem concerning the nature of gravity might have been due to the fact the force has been named, unlike ‘general’ forces, and therefore had become reified in the lexicons of the learners. If the lexical entry was missing the learners adopted a ‘best-fit’ approach; they attempted to analyse the problem using ‘folk definitions’ and use their everyday experience to explain the scenarios.

Once the lexical entries are well defined the learners can begin to progress from bottom-up parsing to a more efficient top-down parsing strategy. Once they are able to use a top-down parsing approach, facilitated by the combining of ‘small’ concepts to form ‘super’ concepts, learners understand the semantics of the question.

If the learners’ conceptions are compared to the tutors, it is apparent that the bottom-up parsing approach adopted by the learners created difficulties. They tended to focus upon the atomic structures of the questions and in a way lost sight of the problem. On the other hand the tutors, who were very familiar with this type of contrived problem, parsed the questions in a top-down fashion and hence did not focus on the wording of the question. They were in some respects at a loss to see what the problem was; to them the solution was obvious.

It seems from this validation process that when a problem is parsed any preconceived notions, including assumptions, are linked to the lexical entry. In the case above, the tutors naturally assumed the box was in a gravitational field and therefore could not remain suspended in mid-air. The learners did not make this assumption automatically and therefore spent time trying to justify how the box could remain in mid-air. The implications of this on their cognitive load are enormous. If the learner has to spend time and hence cognitive resources trying to find a resolution that is unreasonable then it is no wonder they become frustrated and rely upon ‘folk definitions’ of mathematical concepts.

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