Lower secondary school students' knowledge of fractions

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In this paper we present some preliminary data from the ESRC funded ICCAMS project, and compare current Key Stage 3 students' performance on fractions and decimals items with students from 1977. We also present some interview data concerning students' models of fractions, and in particular their use of diagrams to represent part-whole relationships.

Keywords**: Fractions, decimals, models**

Background

Increasing Student Competence and Confidence in Algebra and Multiplicative Structures (ICCAMS) is a 4-year research project funded by the Economic and Social Research Council in the UK (Hodgen et al. 2008). In this paper, we report and discuss early findings of the study regarding students' understanding of fractions. In particular, we compare Key Stage 3 (ages 11-14) students' performance in 2008 and 1977 on items probing their understanding of fractions and decimals. We then discuss some interview responses to a fractions item, with particular reference to their use of diagrams.

Methods and theoretical framework

Phase 1 of the ICCAMS project consists of a large-scale survey of 11-14 years olds' understandings of algebra and multiplicative reasoning in England using three CSMS tests, Algebra, Ratio and Decimals, and an attitudes questionnaire. Items from the Fractions test were added to the Ratio test in the 2008 administration. These tests were carefully designed over the 5-year project starting with diagnostic interviews. (See Hart 1981, for a discussion of the test development.) In Phase 2 of the study we are conducting a collaborative research study with eight teachers extending the investigation to classroom / group settings and examining how assessment can be used to improve attainment and attitudes.

The data in this paper are drawn from the Phase 1 Ratio and Decimals tests and from Phase 2 group interviews.

Participants

In June and July 2008, tests were administered to a sample of approximately 3000 students from 10 schools and approximately 90 classes. We report here on items from the Decimals test and the Ratio test (to which we had appended some fractions items). 2015 students took the Decimals test and 2022 students took the Ratio test. The sample was randomised and drawn from MidYIS, the Middle Years Information System. MidYIS is a value added reporting system provided by Durham University, which is widely used across England (Tymms and Coe 2003). When the cross-sectional survey is completed in 2009 with a further group of 3000 students, the sample will be representative of schools and students in England.

Theoretical framework

There seems to be a widespread consensus among researchers that the rational number construct can usefully be seen in terms of five subconstructs: part-whole relations, ratios, quotients, measures and operations (e.g., Kieren 1980; Behr et al. 1992; Pitkethly and Hunting 1996). Pitkethly and Hunting (1996), in their review of research into the development of early fraction concepts, go on to suggest that the part-whole and ratio subconstructs are fundamental in this development. Part-whole relations are strongly emphasised in the English National Curriculum, and some have argued that this emphasis may be to the detriment of a broader understanding of fraction. Kerslake (1986) suggests that by starting out from the part-whole model, "a major accommodation is required before a fraction can be thought of as a number or as the result of dividing the numerator by the denominator" (p 89). She further states:

> The ready availability of the 'part of a whole' model may itself be the inhibiting feature. If, in thinking of the fraction 3/4, say, the image that immediately springs to mind is that of a circle split into four parts of which three are shaded, then it may prove difficult to adjust to an alternative image of three circles and four people. (p 90)

Nunes (2006) makes a similar point and goes as far as to suggest that the division model (exemplified by, say, 6 children sharing 2 pizzas) chimes better with young children's intuitions about fractions.

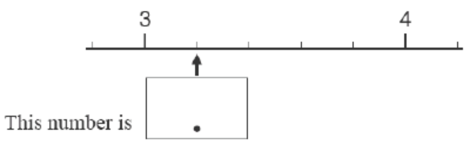
Since the introduction of the National Curriculum in the 1980s, the number line also features strongly in English schools. It is used particularly for learning about operations on whole numbers in the primary school, and on integers in the secondary school. However, as we shall see, our data also suggest it has had a positive impact on the subconstruct of rational number as measure, at least as far as decimal representations are concerned (our interviews seem to suggest that this does not apply to common fractions, although we lack test data on this).

Early Test Analysis: Student performance on fractions and decimals

We note that our early test results should be treated with caution. In particular, we note that the survey is due to be completed in Summer 2009 and that our current sample of students appears to be slightly higher attaining than the general population in England. This early and at this stage tentative analysis suggests that, at age 14, attainment in decimals has risen, is largely unchanged in ratio and has fallen in fractions. The changes in relative performance in decimals and ratio is perhaps unsurprising in that it reflects changes in the balance of the primary and secondary mathematics curriculum. Moreover, the use of decimals generally is far more widespread now than 30 years ago. However, taken as a whole, the data suggest that the well-publicized increases in examination performance in England are not matched by increases in conceptual understanding across mathematics. We emphasize again that this is early analysis and a fuller and more detailed analysis will be published in due course. Further, we note that the patterns across the attainment range, across the age range and across items appear to be rather more complex. The items discussed below have been chosen to be illustrative of student progression and the differential performance of items.

Item 6d (Figure 1) is typical of the broad pattern of attainment in decimals. This item is designed to test students' conceptual understandings of decimal place value in relation to the number line. As can be seen graphically in Figure 1, the item facility has risen considerably from 1977 to 2008 across the 11-14 age range. For example, at age 14 (Year 9), the facility for this item in 2008 was 83% compared to 50% in 1977. Indeed, the current facility of 70% at age 12 (Year 7) is higher than that for age 14 in 1977. One explanation of this is the increased use of the number line in the primary mathematics curriculum (Askew et al. 2002).

Items 12e (Decimals) and F18 (Ratio, originally Fractions) illustrate the difference in performance within and between tests. These items ask *how many* fractions / numbers lie between $\frac{1}{2}$ and $\frac{1}{4}$ and 0.41 and 0.42, respectively. The facilities are shown graphically in Figure 2. As can be seen, there is an improvement in performance on the decimals item, but this is very slight (at Year 9, 23% in 2008 against 21% in 1977). This may be because the item is non-routine and could be said to involve an element of problem solving. Performance on the fractions item has declined (at Year 9, 6% in 2008 against 15% in 1977). This may be because there is now less emphasis on fractions (as opposed to decimals) in the curriculum.



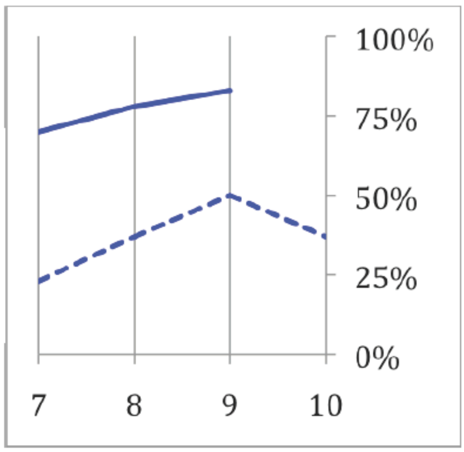
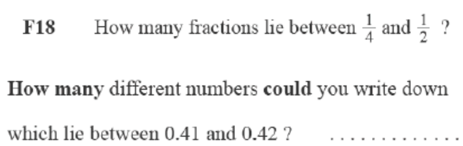


Figure 1: Decimals item 6d. The item is presented alongside several other similar items and students are asked to give their answers as decimals. Facilities are shown for the item in both 2008 [continuous] and 1977 [dotted] for Year 7 to Year 10 (ages 11-15). In 2008 data were not collected for Year 10.



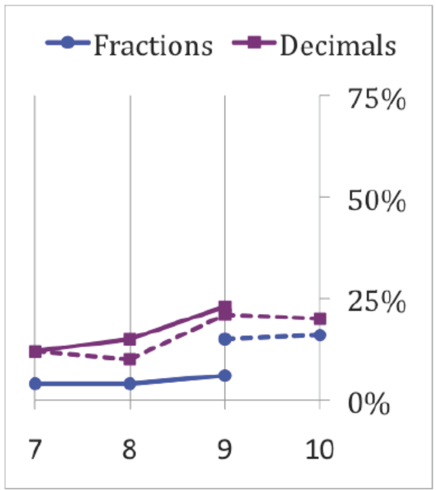


Figure 2: Fractions item 18 and Decimals item 12e. Facilities for items asking how many fractions / numbers lie between $\frac{1}{2}$ and $\frac{1}{4}$ [Fractions; Item 18], and 0.41 and 0.42 [Decimals: Item 12e]. Facilities are shown for both 2008 [continuous] and 1977 [dotted] for Year 7 to Year 10 (ages 11-15). In 2008 data were not collected for Year 10, whilst in 1977, data for the fractions item were collected only for Year 9 and Year 10.

Findings from interviews

We have used item F18 in several open-ended group interviews (of 2, 3 or 4 Year 8 students). The item gave us the opportunity to see what spontaneous models students had for fractions.

One quite common tendency was to think in terms of decimals. Thus one group of students had decided that 1/3 lay between 1/4 and 1/2. This was justified by one student in terms of ‘the bigger the number (denominator), the smaller the fraction’, while another gave this explanation: “A half is 0.5 and a third is 0.3 and a 4th is 0.25 and it’s in between 0.25 and 0.5”.

Not surprisingly, another common tendency was to use a part-whole model to represent fractions, usually by considering parts of a circle (or pizza, etc). A group of two students, R and T, had also decided that 1/3 lay between 1/4 and 1/2. T then suggested 1/5, which R rejected as being too small, because “if you got a circle and split it into quarters, if you split it into 5ths there’s one more to get in there”. This was a nice, grounded explanation, but interestingly involving an imagined rather than an actual drawing. T then suggested 3/5. Asked how we might check this, R suggested “Draw a pie”, which she proceeded to do quite effectively (see Figure 3, below). Using the diagram R was able to reject 3/5 and to decide that “2/5 would be OK”.

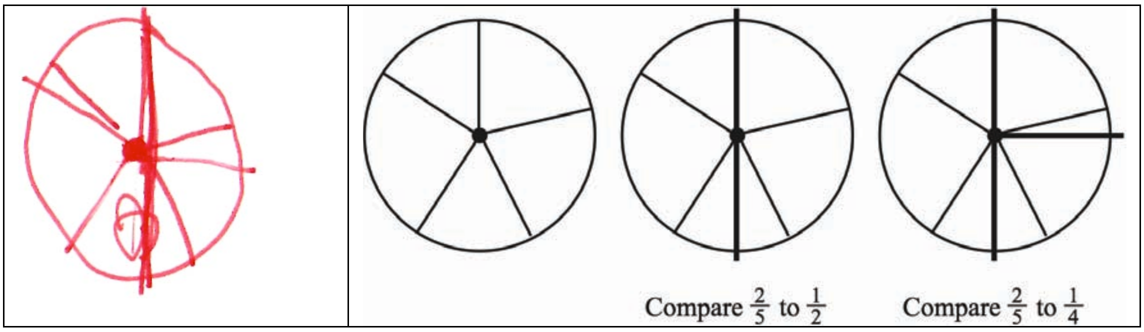


Figure 3. R’s diagram showing 5ths, used to compare 3/5 and 2/5 to 1/2 and 1/4

However, and somewhat to our surprise, R then suggested 3/6 as a possible fraction between 1/4 and 1/2. She proceeded to draw a circle to represent 6ths, which she did in quite a sophisticated way, by drawing diameters through the circle (Figure 4, below). This might be thought to suggest R had some intuitive understanding of how 6ths relate to 1/2, but strangely, she then halved the circle not by using one of her partition lines but by drawing a vertical line which passed through two of the regions representing 6ths. R somehow concluded that 3/6 is smaller than 1/2.

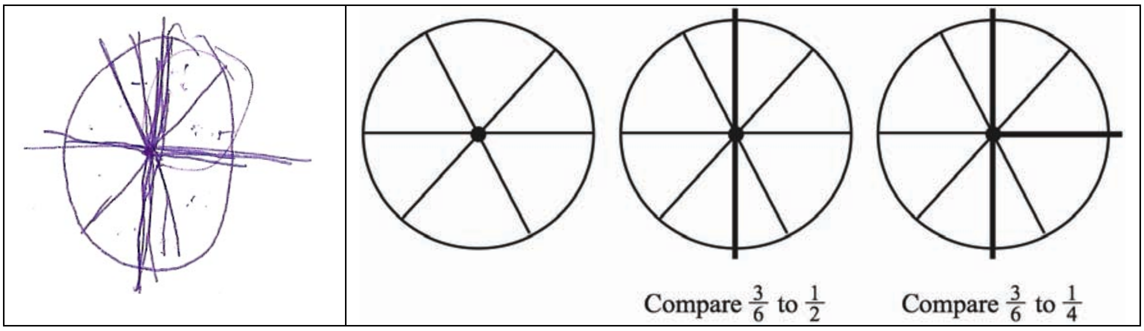


Figure 4. R’s diagram showing 6ths, used to compare 3/6 to 1/2 and 1/4

R then drew another circle to represent 6ths, but this time it was done in a rather short-sighted, step-by step way, resulting in quite irregular sized partitions (Figure 5, below), and leading to R abandoning the drawing.

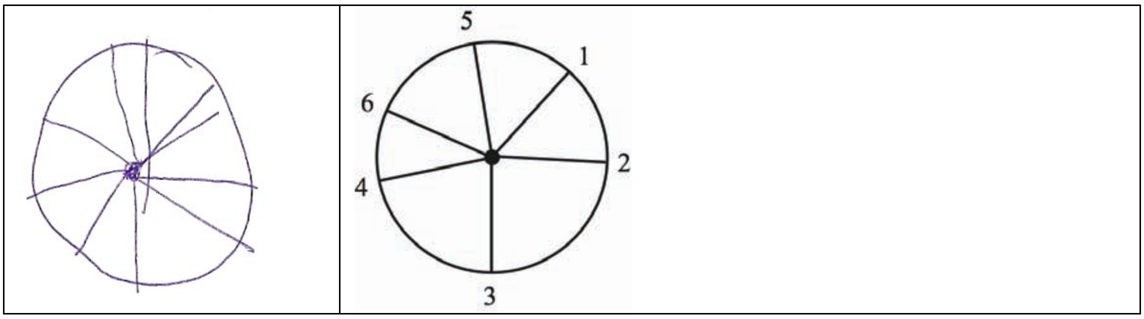


Figure 5. R's second diagram showing 6ths, with the order in which the lines were drawn

T was then asked what he thought and hesitantly replied "I think it would be the same as half, wouldn't it?", whereupon R exclaimed "Yes!". This interchange suggests that R had known that 1/2 and 3/6 are equivalent, but that her use of diagrams had not helped her retrieve this knowledge. There seems to be a paradox here. The diagram is being seen as providing concrete evidence but often it can only be used as an aid to thinking if it is *not* taken 'literally' but merely as a rough representation of an ideal. Put another way, for students to draw effective diagrams, they must be aware in advance of the relationships they are trying to represent.

Figure 6a

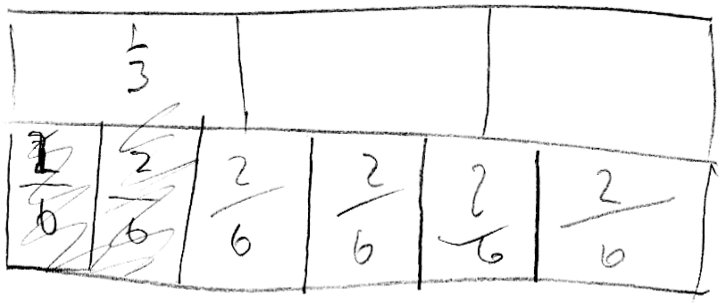


Figure 6b

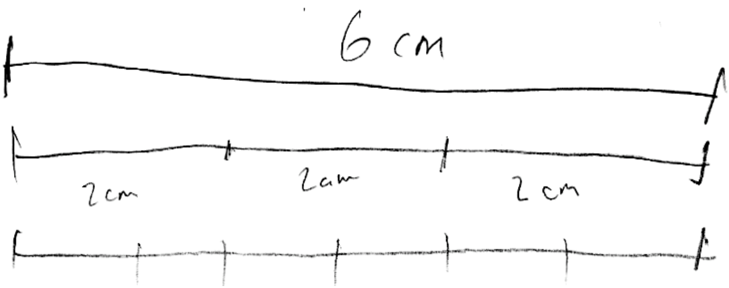


Figure 6. Fraction wall and number line to compare 1/3 to 2/6

With another group of three students we had got on to drawing a fraction wall to represent a whole, halves, 3rds, quarters, 5ths and 6ths. During the course of this one of the students concluded from their drawing that 2/6 was less than a third. We agreed that we needed to be cautious about concluding this as our drawing was not very accurate, and so the interviewer sketched a new wall to show 3rds and asked the students to draw 6ths underneath (Figure 6a, above). Unfortunately, it turned out that the resulting drawing confirmed their misconception. Again, the student who drew this, had proceeded in an empirical, step by step way, when what was needed was the realisation, in advance, that one could extend the partition lines for the thirds, and that two 6ths would fit into one 3rd.

The students suggested that we needed to use a ruler and in response to this the interviewer drew a line, notionally 6 cm long (Figure 6b, above), and a second similar line notionally divided into 2cm lengths to represent 3rds. The students were then asked to mark off 6ths on a third similar line, which this time they were able to do very effectively.

Thus, with the structuring offered by the idea of a ruler, and the conveniently chosen length of 6 cm, the students were this time able to show the equivalent fractions. These observations fit with those of other researchers. Thus, for example, Kerslake (1986) found that all of her interview sample of 12 - 14 year old students could read-off some equivalent fractions when shown ready-made diagrams with identical shaded regions partitioned in different ways. On the other hand, students would draw diagrams to confirm rather than to test their errors (e.g., to show that 3/4 is larger than 4/5, or that 2/3 + 3/4 = 5/7). A similar phenomenon is reported by Herman et al. (2004). This suggests that using one's own diagrams effectively is much more demanding, and more indicative of a sound understanding, than using ready-made diagrams.

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