BSRLM Geometry working group: proof and proving in current classroom materials

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Research across many countries reports that teaching the key ideas of proof and proving to all students is not an easy task. This paper reports on the session of the BSRLM Geometry Working Group which examined current classroom material from the UK with the intention of uncovering the 'opportunities for proof' in geometry that are provided by such material. To carry out such an analysis three analytical frameworks are compared. Two of the analytical frameworks, while placing proof and proving in a wider context of learners' mathematics, may not fully uncover the detail of proof and proving. The third analytical framework, while permitting a detailed analysis of explicit proof and proving, may not fully account for textbooks that devote most space to discussions of proof and proving and/or contain problems that implicitly provoke proof. This comparison reveals some of the complexity of textbook analysis and suggests that further work is needed on a suitable analytical framework.

Keywords: Mathematics; Geometry; Proof; Proving; Textbooks; Secondary school.

Introduction

As the Trends in International Mathematics and Science Study continues to confirm (see, for example, Foxman 1999, Mullis, Martin and Foy 2008), in most countries the textbook remains the primary basis of mathematics instruction. On average, internationally, over 60 percent of teachers report using a textbook as the primary basis of their lessons – with a further 30 percent or more reporting using textbooks as a supplementary resource. In terms of textbook use in England, the latest TIMSS results (see Mullis, Martin and Foy 2008, 288-291) indicate that using a textbook as the primary basis of lessons remains fairly stable at the lower secondary school level. For example, with Year 9 pupils (aged 13-14), over 40 percent of teachers in England report using a textbook as the primary basis of their lessons – with a further 45 percent or more reporting using a textbook as a supplementary resource.

Proof and proving is central to mathematics; yet research across many countries reports that the teaching of the key ideas of proof and proving to all students is not an easy task (for a review see, for example, Mariotti 2007). Given this, it is fitting that proof and proving is the focus of a current ICMI (International Commission on Mathematical Instruction) study (see Hanna and de Villiers 2008). This ICMI study focuses on three major features of 'developmental proof', viz (ibid, p330):

1. Proof and proving in school curricula have the potential to provide a long-term link with the discipline of proof shared by mathematicians.
2. Proof and proving can provide a way of thinking that deepens mathematical understanding and the broader nature of human reasoning.

3. Proof and proving are at once foundational and complex, and should be gradually developed starting in the early grades.

All this indicates that it is worthwhile examining the design of textbooks with a view to uncovering the 'opportunities for proof' in geometry that are provided by current lower secondary school classroom material. The intention of this paper is to complement earlier papers from the BSRLM Geometry Working Group that have reported on the nature of geometrical reasoning (Jones 1998), the forms of classroom tasks that support the proving process (Mogetta, Olivero and Jones 1999), the teaching and learning of proof and proving in geometry (Jones and Rodd 2001), and opportunities provided in contemporary textbooks in England, Scotland and Japan for the development of students' geometrical reasoning (Fujita and Jones 2002).

Research on textbooks

While, as noted above, textbooks remain a feature of almost all mathematics classrooms at the secondary school level, research on their use in the classroom is comparatively limited. Nevertheless, there are a number of relevant studies.

Hanna and de Bruyn (1999), for example, investigated the frequency of items presenting proofs, discussions of proof, and exercises requiring the construction of proofs in a sample of textbooks used in Grade twelve (students age 17-18) in mathematics in Ontario, Canada. Their study revealed that the textbooks were finely attuned to the Ontario Curriculum Guideline and that only in the topic of geometry did the textbooks do a "reasonable job" of providing opportunities to learn proof.

Pepin and Haggarty (2001) report on the use of mathematics textbooks in England, France and Germany. They found that in some textbooks, exercises predominated, with few connections made between the concepts practised - while in others, student exploration, questioning and autonomy were encouraged. Herbel-Eisenmann (2007) and Herbel-Eisenmann and Wagner (2007) report on how mathematics textbooks can 'position' the mathematics learner in relation to classmates and to the world outside of the classroom – and, in particular, what 'authority' is given to student mathematical reasoning and justification.

Vincent and Stacey (2008) examined a selection of three topics in a sample of nine Grade 8 textbooks from four Australian states. They looked at the procedural complexity of problems, the type of solving processes, the degree of repetition, the proportion of 'application' problems, and the proportion of problems requiring deductive reasoning. While reporting considerable differences between textbooks and between topics within textbooks, they conclude that the textbooks analysed featured a relatively high proportion of problems of low procedural complexity, with considerable repetition, and an absence of deductive reasoning.

Fujita and Jones (2002) report on an analysis of the approach to geometrical reasoning presented in best-selling lower secondary school textbooks from England, Scotland and Japan. They report that the textbooks from England and Scotland were primarily designed around a set of exercises - with mathematical theorems stated, rather than being developed or proved. In contrast, the selected Japanese textbooks attempted to develop students' deductive reasoning through teaching 'proof' using various approaches.

All this indicates that it is worth continuing to examine the 'opportunities for proof' in geometry that are provided by current lower secondary school classroom material as it is clear that there are likely to be different ways of analysing the complex relationship between the tasks presented in the textbooks and the impact that this has on student learning.

Framework for analysing proof and proving in textbooks

The approach used by Fujita and Jones (2002, 2003) to analyse textbooks was derived from the work of Valverde et al (2002). The technique is to section pages of the textbook into relatively coherent 'blocks' and then to code of each 'block' in terms of content, performance expectations and perspectives. The codes used for the analysis are shown in Appendix A.

Vincent and Stacey (2008) make use of the 'Coding Manual' developed for the 1999 TIMSS Video Study (see LessonLab 2003). Here proof was defined as "the process of establishing the validity of a statement, especially by definition from other statements in accordance with principles of reasoning"; verification was defined as "the act or process of ascertaining the truth or correctness of a rule"; and derivation was defined as "a sequence of statements showing that a result is the necessary consequence of previously accepted statements" (ibid, 66). The approach is that to qualify as a proof, verification or derivation, the target result "must apply to a class of problems (for example, proof of the Pythagorean theorem) rather than a single problem, must be non-numeric, and must be arrived at through deductive reasoning" (Vincent and Stacey 2008, 90).

Hanna and Bruyn (1999), by focussing solely on proof and proving, adopted a framework where sections of a textbook were classified into three categories: (1) non-proof, (2) discussion of proof, and (3) proof. Non-proof items were those that had nothing to do with proving, such as calculating unknown angles in a geometric figure provided in the text. The statement of a theorem without proof or definition of terms was also considered a non-proof. Discussion items were those which discuss the creation of a proof, or provide guidance in how to go about it. Proof items included any section which provided a full or partial proof, such as the proof of a trigonometric identity or a geometry theorem, and any exercise which included the imperatives 'prove' or 'show'. Proof items were further divided into two broad categories: direct proof (proven through a sequence of deductions) and indirect proof (showing that if the proposition were false, it would lead to a contradiction). Hanna and Bruyn then broke down direct proofs into five further sub-categories: basic (just the direct proof), by analysis (for example, working backwards from the conclusion), existence or construction proof, proof by induction, and miscellaneous.

Discussion

The advantage of the approaches adopted by Fujita and Jones (2002, 2003) and by Vincent and Stacey (2008) is that geometrical reasoning is placed in a wider context of students work in mathematics. The disadvantage is that any 'justifying and proving' (in the case of Fujita and Jones) or 'proof, verification or derivation (PVD)' (in the case of Vincent and Stacey) is coded with the single code (of '2.4.5' - see Appendix A – in the case of Fujita and Jones, and PVD in the case of Vincent and Stacey) and this might mask differences in approaches to proof and proving. Nevertheless, the analysis framework adopted by Fujita and Jones does provide a means of differentiating between other aspects of proof and proving such as 'developing notation and vocabulary' (code 2.4.1), developing algorithms (code 2.4.2), generalising (code 2.4.3), and conjecturing and discovering (code 2.4.4).

Hanna and Bruyn (1999), in their analysis, differentiated between non-proof, discussion of proof, and explicit proof. The advantage of this approach is that, through focussing solely on proof and proving, a more sophisticated characterisation of proving is possible. A disadvantage is that the greatest sophistication in the analysis framework is in the component most developed is that of 'proof' in that this is not only further divided into two broad categories - direct proof and indirect proof – but then the category of direct proof is further divided into five sub-categories: basic, by analysis, existence or construction proof, proof by induction, and miscellaneous. This might be fine when textbooks contain 'direct' proofs, but it may not fully account for textbooks that devote most space to 'discussion of proof' and that, perhaps as a consequence, do not dictate the form of proof in the student version of the textbook but rather provide suggestions to the teacher in the 'teacher version' of the textbook.

Concluding comments

In this paper, three analytical frameworks for uncovering the 'opportunities for proof' in geometry, as presented in school textbooks, are compared. Two of the analytical frameworks, those of Fujita and Jones (2002, 2003) and of Vincent and Stacey (2008), while placing proof and proving in a wider context of learners' mathematics, may not fully uncover the detail of proof and proving. The third analytical framework, by Hanna and Bruyn (1999), while permitting a detailed analysis of explicit proof and proving, may not fully account for textbooks that devote most space to discussions of proof and proving and/or contain problems that implicitly provoke proof. This comparison reveals some of the complexity of textbook analysis and suggests that further work is needed on a suitable analytical framework.

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BSRLM geometry working group

The BSRLM geometry working group focuses on the teaching and learning of geometrical ideas in its widest sense. Suggestions of topics for discussion are always welcome. The group is open to all.

Appendix A: analytical framework from Fujita and Jones (2003)

Block type	Content (subject matter topic)	Performance Expectations	Perspective
1 Central instructional narrative 2 Related instructional narrative 3 Unrelated instructional narrative 4 Graphic (those directly related narrative) 5 Graphic (those not directly related narrative) 6 Question 7 Exercise Set 8 Suggested activities 9 Worked examples 10 Others	1.1. Geometry: Position, visualisation, and shape 1.1.1. Two-dimensional geometry: Co-ordinate geometry 1.1.2. Two-dimensional geometry: Basics (point, line, and angles) 1.1.3. Two-dimensional geometry: Polygons and circles 1.1.4. Three-dimensional geometry 1.1.5. Vectors 1.2. Geometry: Symmetry, congruence, and similarity 1.2.1. Transformation 1.2.2. Symmetry 1.2.3. Congruence 1.2.4. Similarity 1.2.5. Constructions using straightedge and compass 1.3. Measurement 1.3.1. Perimeter, area, and volume 1.3.2. Angle and bearing	2.1. Knowing 2.1.1. Representing 2.1.2. Recognising equivalents 2.1.3. Recalling properties and theorems 2.1.4. Consolidating notation and vocabulary 2.1.5. Recognising aims of lessons 2.2. Using routine procedures 2.2.1. Using equipment 2.2.2. Performing routine procedures 2.2.3. Using more complex procedures 2.3. Investigating and problem solving 2.3.1. Formulating and clarifying problems 2.3.2. Developing strategy 2.3.3. Solving 2.3.4. Predicting 2.3.5. Verifying 2.4. Mathematical reasoning 2.4.1. Developing notation and vocabulary (proof) 2.4.2. Developing algorithms 2.4.3. Generalising 2.4.4. Conjecturing and discovering 2.4.5. Justifying and proving 2.4.6. Axiomatising 2.5. Communicating 2.5.1. Using vocabulary and notation 2.5.2. Relating representations 2.5.3. Describing/discussion 2.5.4. Critiquing	3.1. Attitude toward science, mathematics, and technology 3.2. Careers involving in science, mathematics, and technology 3.2.1. Promoting careers in science, mathematics, and technology 3.2.2. Promoting the importance of science, mathematics, and technology in non-technical careers 3.3. Participation in science and mathematics by underrepresented groups 3.4. Science, mathematics and technology to increase interest 3.5. Scientific and mathematical habits of mind